M321 8, 9 and 10

THE OPEN UNIVERSITY

Mathematics: A Third Level Course
Partial Differential Equations of Applied Mathematics Units 8, 9 and 10

Stability
Green's Functions I
Green's Functions II

TO

HIS GRACE THE DUKE OF NEWCASTLE, K.G.

Lord Lieutenant of the County of Nottingham;

VICE PRESIDENT OF THE ROYAL SOCIETY OF LITERATURE,

&c. &c. &c.

MY LORD DUKE,

I AVAIL myself of your GRACE'S kind permission to introduce the following Essay to the notice of the Public, under your high auspices; and I deem myself singularly fortunate, that my first attempt to illustrate some of the most interesting phenomena of nature, should make its appearance under the patronage of a NOBLEMAN, who has always evinced a most lively interest in the promotion of Science and Literature, and particularly in the County over which he so eminently presides.

I have the honor,

MY LORD DUKE,

To be your Grace's most obedient

and grateful Servant,

GEORGE GREEN.

Sneinton, near Nottingham, March 29th, 1828.

AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

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1828.

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Mathematics: A Third Level Course

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STABILITY
GREEN'S FUNCTIONS I
GREEN'S FUNCTIONS II

Prepared by the Course Team

The Open University Press

Cover Illustration

The cover of this volume features extracts from *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* by G. Green (Nottingham, 1828), courtesy of University of London Library, the De Morgan collection. This is the paper in which potential functions (now known as *Green's functions*) were first introduced: these functions are studied in *Units 9 and 10*. The theorem on the back cover is a variant of the result that we met in *Unit 3, Elliptic and Parabolic Equations* under the name of *Green's Theorem*. Note the use of the symbol δ for the Laplacian operator ∇^2.

George Green was born in 1793 at Sneinton, near Nottingham. He was almost entirely self-taught as a mathematician and it was not until 1833 that he took up residence at Gonville and Caius College, Cambridge. Three years later he graduated as Fourth Wrangler and in 1839 he was elected a Fellow of his College. He did not enjoy the status of Fellow for long, however, as he died in 1841.

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1.1

Unit 8 Finite-Difference Methods II: Stability

Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford. 1971).

H. F. Weinberger, *A First Course in Partial Differential Equations* (Blaisdell, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as *S* and *W* respectively.

Unit 8 is based on *S*: Chapter 3, pages 58 to 72.

Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics*. References to Open University courses in mathematics take the form:

> *Unit M100 13, Integration II* for the Mathematics Foundation Course.
> *Unit M201 23, The Wave Equation* for the Linear Mathematics Course.

8.0 INTRODUCTION

In *Unit 5, Initial Value Problems* we showed how to construct both explicit and implicit finite-difference schemes for initial value problems, how to determine their local accuracy and how to solve the resulting equations in the implicit case. We demonstrated that good local accuracy, measured in terms of the local truncation error, is not sufficient to guarantee a successful method, and concluded that for parabolic equations the mesh ratio is often an important parameter in this respect. In this unit we look more closely at the theory of finite-difference methods applied to initial value problems.

First we introduce the idea of convergence of a finite-difference scheme. We say that a scheme is *convergent* if, as the mesh spacings are reduced, the finite-difference solution gets steadily closer to the true solution of the differential equation either at a fixed point, or for all points along the furthest time level under consideration. Convergence is our prime objective when using a finite-difference scheme, and in practice we might deliberately compute successive approximate solutions, using smaller and smaller intervals, until we get results to a required precision. We shall see that for this to happen the local truncation error, defined in *Unit 5*, must tend to zero as the mesh spacings approach zero.

Additionally we need the finite-difference scheme to be *stable*; that is, small local errors of any kind (truncation or rounding errors) should not grow unboundedly as the step-by-step computation progresses. Thus convergence depends on both the finite-difference scheme *and* the partial differential equation whereas stability depends only on the finite-difference scheme

In Section 8.2 we shall consider sc me methods for establishing convergence in fairly simple cases. These methods are also useful for finding the *rate* of convergence of a scheme. A similar process was followed in the case of ordinary differential equations in *Unit M201 21, Numerical Solution of Differential Equations.* We shall then introduce an important theorem, in Section 8.3, which relates convergence to stability and the behaviour of the local truncation error, and enables convergence to be established more easily and in more general cases. Finally, we concentrate on stability in Section 8.4.

Notations often differ in the literature; we shall use the following which, for the most part, agree with *S*.

$U_{i,j}$ is the true solution $U(ih, jk)$ of the differential equation at the mesh point $(x, t) = (ih, jk)$.

$u_{i,j}$ is the true (or exact) solution of the finite-difference scheme at (ih, jk).

$\bar{u}_{i,j}$ is the approximate solution of the finite-difference scheme at (ih, jk) obtained from a computation.

Thus, $U_{i,j}$ contains no error, $u_{i,j}$ includes truncation errors and $\bar{u}_{i,j}$ has both truncation and rounding errors.

8.1 THE BASIC IDEAS OF CONVERGENCE AND STABILITY

To illustrate the ideas of convergence and stability we shall consider the simple explicit scheme given by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \tag{1}$$

as applied to the initial-boundary value problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad x \in [0, 1],\ t \in R^+,$$

$$U(x, 0) = f(x) \qquad x \in R, [0, 1],$$

$$U(0, t) = U(1, t) = 0 \quad t \in R^+.$$

Suppose that we know the solution u at all mesh points up to and including those at time level j. We can evaluate an approximation $u_{i,j+1}$ to $U_{i,j+1}$ from Equation (1) by

$$u_{i,j+1} = u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \qquad i = 1, 2, \ldots, N-1, \tag{2}$$

$$u_{0,j+1} = u_{N,j+1} = 0,$$

where $r = k/h^2$ and $Nh = 1$. For brevity, we write this as

$$\mathbf{u}_{j+1} = L\mathbf{u}_j, \tag{3}$$

where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})$ and L is a (linear) operator $R^{N-1} \longrightarrow R^{N-1}$ which can be represented by the matrix

$$\begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & r & 1-2r \end{bmatrix}.$$

In *Unit 5* (Section 5.5) we defined the local truncation error $T_{i,j}$ for the explicit scheme (1) as

$$T_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k} - \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}.$$

We also introduced the quantity $T^*_{i,j+1}$ given by

$$\begin{aligned} T^*_{i,j+1} &= kT_{i,j} \\ &= U_{i,j+1} - U_{i,j} - r(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}), \end{aligned} \tag{4}$$

i.e.

$$\mathbf{U}_{j+1} = L\mathbf{U}_j + \mathbf{T}^*_{j+1}. \tag{5}$$

We now illustrate the assertion in *Unit 5* that the local truncation errors can accumulate as the step-by-step process progresses. We can write Equation (5), for the first time step, as

$$\mathbf{U}_1 = L\mathbf{U}_0 + \mathbf{T}^*_1. \tag{6}$$

Repeating the procedure for the second time step, we get

$$\begin{aligned} \mathbf{U}_2 &= L\mathbf{U}_1 + \mathbf{T}^*_2 \\ &= L^2\mathbf{U}_0 + L\mathbf{T}^*_1 + \mathbf{T}^*_2 \qquad \text{by Equation (6).} \end{aligned}$$

As usual $L^2 = L \circ L$ means "apply the operator L twice". Repeating the process again gives

$$\begin{aligned} \mathbf{U}_3 &= L\mathbf{U}_2 + \mathbf{T}^*_3 \\ &= L^3\mathbf{U}_0 + L^2\mathbf{T}^*_1 + L\mathbf{T}^*_2 + \mathbf{T}^*_3, \end{aligned}$$

and we may deduce the general case,

$$\mathbf{U}_n = L^n\mathbf{U}_0 + \sum_{j=1}^{n} L^{n-j}\mathbf{T}_j^* \qquad (L^0 = 1), \tag{7}$$

by induction. This equation is exact; it relates the true solution of the problem to the initial data $\mathbf{U}_0$ and the local truncation errors.

When we solve Equation (1) exactly (without rounding error), we obtain

$$\mathbf{u}_1 = L\mathbf{u}_0,$$

$$\mathbf{u}_2 = L\mathbf{u}_1 = L^2\mathbf{u}_0,$$

$$\vdots$$

$$\mathbf{u}_n = L^n\mathbf{u}_0,$$

and since the initial data are given by the true solution at time level zero, that is $\mathbf{u}_0 = \mathbf{U}_0$, we have

$$\mathbf{u}_n = L^n\mathbf{U}_0. \tag{8}$$

Thus we can obtain an expression for the **global truncation error** $\mathbf{U}_n - \mathbf{u}_n$ in the solution to the finite-difference scheme. From Equations (7) and (8) we obtain, for the explicit scheme (1),

$$\mathbf{U}_n - \mathbf{u}_n = \sum_{j=1}^{n} L^{n-j}\mathbf{T}_j^*. \tag{9}$$

Whether or not these local errors accumulate numerically will depend on the effect of the operator L and the nature of the $T_{i,j}^*$. The local truncation error $T_{i,j}$ may be expressed (*Unit 5*, Section 5.5) in terms of derivatives of the true solution and we can infer the nature of the $T_{i,j}^*$.

We now concentrate on the rounding errors in the computed solution of Equation (1). We assume first that the initial data $u_{i,0} = U_{i,0}$ cannot be stored exactly, and so the computer begins with an approximation $\bar{u}_{i,0}$ to $u_{i,0}$ satisfying

$$\bar{\mathbf{u}}_0 = \mathbf{u}_0 + \mathbf{R}_0,$$

where $R_{i,0}$ is the initial rounding error. Since there will be a rounding error introduced at each stage of the computation, the computer will evaluate $\bar{\mathbf{u}}_{j+1}$ as

$$\bar{\mathbf{u}}_{j+1} = L\bar{\mathbf{u}}_j + \mathbf{R}_{j+1}. \tag{10}$$

Thus at the first step we compute

$$\bar{\mathbf{u}}_1 = L\bar{\mathbf{u}}_0 + \mathbf{R}_1 = L\mathbf{u}_0 + L\mathbf{R}_0 + \mathbf{R}_1,$$

and in general

$$\bar{\mathbf{u}}_n = L^n\mathbf{u}_0 + \sum_{j=0}^{n} L^{n-j}\mathbf{R}_j$$

$$= L^n\mathbf{U}_0 + \sum_{j=0}^{n} L^{n-j}\mathbf{R}_j. \tag{11}$$

The difference between the computed solution $\bar{u}$ and the true solution U of the differential equation can be obtained from Equations (7) and (11) and is given by

$$\mathbf{U}_n - \bar{\mathbf{u}}_n = \sum_{j=0}^{n} L^{n-j}(\mathbf{T}_j^* - \mathbf{R}_j), \tag{12}$$

where for convenience we have set $\mathbf{T}_0^* = \mathbf{0}$. Thus, in general, both the $T_{i,j}^*$ and the $R_{i,j}$ can cause a numerical accumulation in the total error. However, in circumstances where the error due to the $T_{i,j}^*$ does not grow (see, for example, the solution to SAQ 15 in *Unit 5*) it is still possible for the error due to the $R_{i,j}$ to grow numerically.

Equation (12) is the expression for the global, or total, error of the explicit scheme (1). In practice we cannot calculate the right-hand side of this equation since the $T_{i,j}$

and hence the $T^*_{i,j}$ depend on derivatives of the true solution which we do not know, and the $R_{i,j}$ depend almost randomly on the computing device we use. Our best bet therefore is to investigate the nature of the operator L to find those conditions under which the right-hand side of Equation (12) remains manageable in the sense that it is "bounded". Now the essence of stability is that there should be a limit to the amount by which any error can be amplified in the numerical procedure. We see that the right-hand side of Equation (12) is a sum of error terms $\mathbf{T}^*_j - \mathbf{R}_j$ operated on by some L^p. We would hope that the magnification of any error is limited, i.e. that there is a real number K such that

$$\|L^p\mathbf{E}_j\| \leqslant K\|\mathbf{E}_j\| \tag{13}$$

for $p = 0, 1, 2, \ldots$, and all local errors $\mathbf{E}_j$, where $\|\mathbf{E}_j\| = \max\limits_{1<i<N}|E_{i,j}|$. We now say that the scheme (1) is **stable** if there exist h_0 and k_0 such that (13) holds for all p, all local errors $\mathbf{E}_j$ and all $h < h_0, k < k_0$.

Before taking this idea any further we shall introduce the idea of convergence. An ideal situation for our numerical scheme would be for the computed solution $\bar{u}$ to approach the true solution U along some fixed time level as we steadily reduce the mesh spacings to zero. Applied to our explicit scheme (1) this would clearly require that the sum on the right-hand side of Equation (12) should approach zero as both h and k approach zero. Expanding Equation (12) as

$$\mathbf{U}_n - \bar{\mathbf{u}}_n = \sum_{j=0}^{n} L^{n-j}\mathbf{T}^*_j - \sum_{j=0}^{n} L^{n-j}\mathbf{R}_j$$

clearly reveals the fact that we can never be assured that $\|\mathbf{U}_n - \bar{\mathbf{u}}_n\|$ approaches zero as $h, k \rightsquigarrow 0$ because the second sum depends not only on h and k but also on the computing device. It is only when $R_{i,j} = 0$ for all i, j that convergence in the above sense can be obtained. We therefore have to amend our ideas, and we now *define* a finite-difference scheme as **convergent** when the true solution u of the scheme (without rounding errors) tends to the true solution U of the differential equation along some *fixed* time level as we steadily reduce the mesh spacings h and k to zero. Since we look at a fixed time level, t say, the fact that h and k tend to zero means that $n\ (= t/k)$, the number of time steps required to reach the fixed time level, increases unboundedly.

It follows from Equation (9) that to prove convergence we must impose conditions on L and $T^*_{i,j}$ so that

$$\lim_{h,k \rightsquigarrow 0} \sum_{j=1}^{n} L^{n-j}\mathbf{T}^*_j = \mathbf{0}. \tag{14}$$

Now, recall that by definition

$$kT_{i,j} = T^*_{i,j+1}$$

for our explicit scheme, so we want

$$\lim_{h,k \rightsquigarrow 0} k \sum_{j=0}^{n-1} L^{n-j-1}\mathbf{T}_j = \mathbf{0}.$$

Now by the generalized triangle inequality we can write

$$\left\| k \sum_{j=0}^{n-1} L^{n-j-1}\mathbf{T}_j \right\| \leqslant k \sum_{j=0}^{n-1} \|L^{n-j-1}\mathbf{T}_j\| \tag{15}$$

and if our scheme is stable, we obtain, by Equation (13), the result that

$$\left\| k \sum_{j=0}^{n-1} L^{n-j-1}\mathbf{T}_j \right\| \leqslant k \sum_{j=0}^{n-1} K\|\mathbf{T}_j\|. \tag{16}$$

If we let M denote the maximum absolute value of the $T_{i,j}$ then Equation (16) becomes

$$\left\| k \sum_{j=0}^{n-1} L^{n-j-1}\mathbf{T}_j \right\| \leqslant kn\, KM.$$

However, $kn = t$ and so this inequality becomes

$$\left\| k \sum_{j=0}^{n-1} L^{n-j-1} \mathbf{T}_j \right\| \leqslant tKM.$$

Now, if we have the additional condition that

$$\lim_{h,k \to 0} T_{i,j} = 0 \qquad \text{for all } i, j,$$

then certainly M also approaches zero. That is, the quantity tKM approaches zero, and so the left-hand side of Equation (15) approaches zero and the scheme is convergent. This latter condition, that the local truncation error $T_{i,j}$ (or $kT^*_{i,j+1}$) approaches zero as the mesh spacings h and k tend to zero, is known as **consistency** (or **compatibility**). We can now say that if our explicit scheme (1) is consistent then stability guarantees convergence. This relationship holds for other schemes and forms the basis of an important theorem which we shall meet in Section 8.3.

Although S does not consider local truncation errors in his descriptive treatment of stability, we have illustrated both here and in *Unit 5* that they play an important part in this subject. As a result there is a difference of emphasis between S and what has been said so far in the correspondence text. Therefore, we are not asking you to read *S: page 55, line 1 to page 58, line 13*.

8.2 CONVERGENCE IN SIMPLE CASES

8.2.1 The Explicit Formula

READ S. the section entitled **"Analytical treatment of convergence"**, *page 58, line 14 to page 60, line 18.*

Notes

(i) ***S**: page 58, line −13*
e is called the global truncation error in our terminology.

(ii) ***S**: page 59, lines 4 to 12*
These equations are exact using the "remainder" form of Taylor's Theorem which states:

> If u is $n + 1$ times differentiable on the interval $[x, x + h]$ then there is a $\theta \in (0, 1)$ such that
>
> $$u(x + h) = u(x) + hu'(x) + \cdots + \frac{h^n}{n!}u^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}u^{(n+1)}(x + \theta h).$$
>
> (If $h < 0$, then $[x, x + h]$ is replaced by $[x + h, x]$.)

This form of Taylor's Theorem is a stronger result than the one we quoted in *Unit 5*.

In obtaining Equation (3.4) from the equations on *lines 5, 6 and 7* in ***S**: page 59*, use has been made of the Intermediate Value Theorem which states:

> If a function f is differentiable on the interval $[a, b]$, then for any real number y between $f'(a)$ and $f'(b)$ there is at least one point $x_0 \in (a, b)$ such that $f'(x_0) = y$.
>
> In particular, there exists $x_0 \in (a, b)$ such that
>
> $$f'(x_0) = \tfrac{1}{2}[f'(a) + f'(b)].$$

Hence

$$\frac{1}{2}\left\{\frac{\partial^2 U}{\partial x^2}(x_i + \theta_1 h, t_j) + \frac{\partial^2 U}{\partial x^2}(x_i - \theta_2 h, t_j)\right\} = \frac{\partial^2 U}{\partial x^2}(x_i + \theta_4 h, t_j)$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$, for some θ_4 such that $-1 < \theta_4 < 1$.

For further details of these theorems and related topics, see Appendix 1 of Kreider, Kuller, Ostberg and Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

(iii) ***S**. page 59, line −1*
The finite-difference scheme uses the same (known) initial data as the differential equation. Therefore, $u_{i,0} = U_{i,0}$ and so $e_{i,0} = 0$ for each i.

General Comment

We can relate the discussion in ***S*** to what we know about local truncation errors. We notice that Equation (3.3) on *page 59* contains the expression

$$U_{i,j+1} - U_{i,j} + r(2U_{i,j} - U_{i-1,j} - U_{i+1,j})$$

which is $T^*_{i,j+1}$ by definition. In this case $T^*_{i,j+1} = kT_{i,j}$, where $T_{i,j}$ is the local truncation error of the explicit scheme. Therefore, we may write

$$e_{i,j+1} = re_{i-1,j} + (1 - 2r)e_{i,j} + re_{i+1,j} + kT_{i,j}.$$

(Compare this with Equation (3.4) in ***S***.)

Let M be the maximum value of the modulus of the $T_{i,j}$ for all i and j; then

$$|e_{i,j+1}| \leqslant r|e_{i-1,j}| + (1-2r)|e_{i,j}| + r|e_{i+1,j}| + kM$$

which is the expression given in *S: page 59, line −8.*

In *Unit 5*, we showed that

$$T_{i,j} = O(h^2) + O(k)$$

and hence $T_{i,j}$ approaches 0 as $h, k \rightsquigarrow 0$. Therefore, M approaches 0 as $h, k \rightsquigarrow 0$, and we conclude the convergence proof in a manner similar to that given at the top of *S: page 60.*

Similar techniques can be used for a simple explicit method applied in a hyperbolic case. This is the basis of the following SAQ.

SAQ 1

Let U be the solution of the differential equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} \qquad x \in R,\ t > 0,$$

subject to the initial conditions

$$U(x, 0) = f(x) \qquad x \in R,$$

$$\frac{\partial U}{\partial t}(x, 0) = g(x) \qquad x \in R.$$

The differential equation is approximated by the explicit finite-difference scheme

$$u_{i,j+1} = u_{i+1,j} - u_{i,j-1} + u_{i-1,j},$$

where the mesh ratio $p^2 = k^2/h^2 = 1$, and the derivative initial condition is approximated by the forward-difference formula

$$\frac{\partial U_{i,0}}{\partial t} \simeq \frac{u_{i,1} - u_{i,0}}{k}.$$

(a) Show that $e_{i,1} = O(k^2)$

where $e_{i,j} = U_{i,j} - u_{i,j}$.

(b) Show that $e_{i,j+1} = e_{i+1,j} - e_{i,j-1} + e_{i-1,j}$,

when U is infinitely differentiable.

(c) Express $e_{i,j+1}$ as a sum of errors on the zeroth and first time levels, and deduce that u converges to U as k tends to zero.

(Solution on p. 25.)

8.2.2 The Courant–Friedrichs–Lewy Condition for Hyperbolic Equations

In this section we explore by means of SAQs how the characteristics of a hyperbolic partial differential equation can help to determine whether a given finite-difference scheme for the equation is satisfactory. Throughout this section we shall refer to the following pure initial value problem and finite-difference scheme.

Problem A

Solve

$$\frac{\partial^2 U}{\partial t^2} - c^2\frac{\partial^2 U}{\partial x^2} = 0 \qquad x \in R,\ t > 0,$$

$$U(x, 0) = f(x) \qquad x \in R,$$

$$\frac{\partial U}{\partial t}(x, 0) = g(x) \qquad x \in R,$$

where c is a positive constant.

Scheme B

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = p^2c^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \qquad i \in Z,\ j \in Z^+,$$

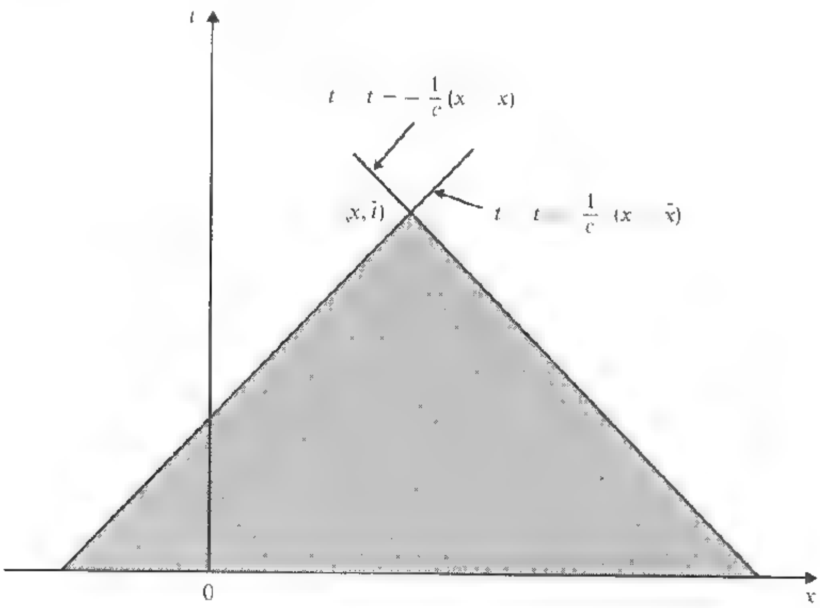
where the mesh ratio p^2 is given by $p = k/h$.

We have seen in *Unit 2, Classification and Characteristics* that the characteristics of Problem A which pass through the point $(\bar{x}, \bar{t})$, $\bar{t} > 0$, are

$$t - \bar{t} = \pm\frac{1}{c}(x - \bar{x}).$$

The *domain of dependence* of the point $(\bar{x}, \bar{t})$, $\bar{t} > 0$, for Problem A is the shaded area in the diagram, specified by

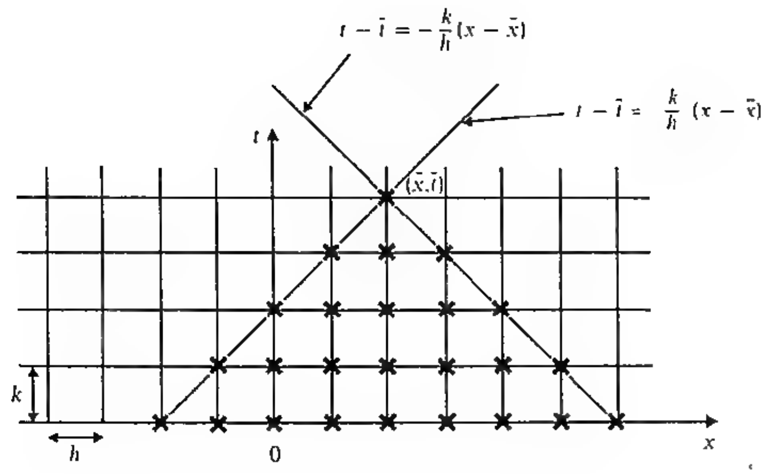
$$\left\{(x, t) : t \geqslant 0,\quad t \leqslant \bar{t} - \frac{1}{c}(x - \bar{x}),\quad t \leqslant \bar{t} + \frac{1}{c}(x - \bar{x})\right\}.$$



We define the **numerical domain of dependence** of a point $(\bar{x}, \bar{t})$, for a given finite-difference scheme, as the set of mesh points whose *numerical domains of influence* (*Unit 5*, Section 5.2) include $(\bar{x}, \bar{t})$. Thus the numerical domain of dependence of a general mesh point $(\bar{x}, \bar{t}) = (\bar{i}h, \bar{j}k)$ for Scheme B is given by

$$\{(ih, jk) : j \geqslant 0,\quad j \leqslant \bar{j} - (i - \bar{i}),\quad j \leqslant \bar{j} + (i - \bar{i})\},$$

and is shown by crosses in the following diagram.



SAQ 2

On what part of the given initial data do the following solutions at $(\bar{x}, \bar{t})$ depend?

(a) the solution to Problem A

(b) the solution to the numerical Scheme B applied to Problem A

(Solution on p.26.)

SAQ 3

Sketch a diagram showing the domain of dependence of a point $(\bar{x}, \bar{t})$ for Problem A. On the same diagram sketch the numerical domain of dependence of the same point $(\bar{x}, \bar{t})$ for Scheme B assuming that $p > 1/c$. What happens to the relative positions of the two domains if the mesh ratio p is altered so that $p < 1/c$?

Under what conditions are these domains of dependence the same?

(Solution on p. 26.)

SAQ 4

If the numerical domain of dependence of a given point $(\bar{x}, \bar{t})$ lies within the domain of dependence of the same point, what happens to the analytical and numerical solutions at $(\bar{x}, \bar{t})$ if changes are made to the initial data at points along the initial line which are:

(i) in both domains of dependence;

(ii) in the domain of dependence but not in the numerical domain of dependence?

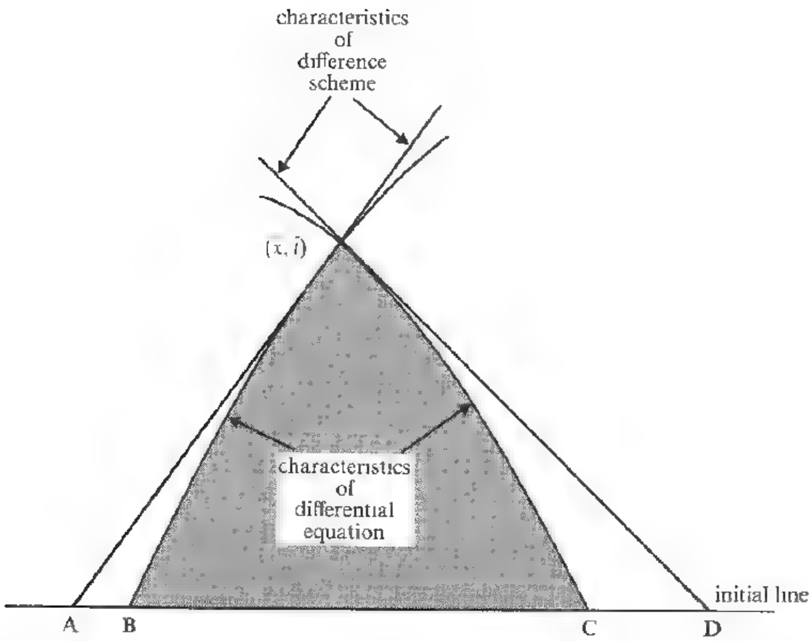
(Solution on p. 27.)

SAQ 5

What general condition must be placed on the mesh ratio for a finite-difference scheme applied to a hyperbolic equation to ensure that all the initial data are used? Apply this condition to Scheme B when it is employed to solve Problem A.

(Solution on p. 27.)

We have seen that the domain of dependence of a point in the solution domain of the difference equation must not lie inside the domain of dependence of the same point in the solution domain of the differential equation. We have not, however, investigated the case when the numerical domain of dependence of a point contains properly the domain of dependence of that point. We now quote a result for this case. It has been shown, by Courant, Friedrichs and Lewy, that the contribution to the solution of the difference equation at the point $(\bar{x}, \bar{t})$ from the extra initial data outside the domain of dependence (along AB and CD in the next diagram) tends to zero as $h, k \to 0$ keeping the mesh ratio p^2 constant, where $(\bar{x}, \bar{t})$ remains a fixed point.



The conclusion is the Courant Friedrichs Lewy condition (or just the **C.F.L. condition**) which states that a necessary and sufficient condition for the convergence of a difference scheme for a hyperbolic equation is that the numerical domain of dependence of a point must include the domain of dependence of the same point.

SAQ 6

Find the C.F.L. condition for the difference scheme

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = p^2c^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

when applied to the hyperbolic equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$$

where c is a positive constant.

(Solution on p. 27.)

8.3 CONSISTENCY AND LAX'S THEOREM

The local truncation error gives us a measure of the local accuracy of a finite-difference replacement of a given partial differential equation. We might expect that if we *refine* the mesh, that is, make the mesh spacings h and k smaller, we would reduce the local truncation error and would have a more accurate scheme In particular we would like to have

$$\lim_{h,k \to 0} T_{i,j} = 0.$$

If this condition holds then the finite-difference scheme is *consistent* (or *compatible*) with the given partial differential equation. Unfortunately, this is not always the case unconditionally, as the following SAQ illustrates.

SAQ 7

Show that Du Fort and Frankel's three-level explicit scheme

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i+1,j} - u_{i,j+1} - u_{i,j-1} + u_{i-1,j}}{h^2}$$

has a local truncation error given by

$$T_{i,j} = \frac{k^2}{6}\frac{\partial^3 U_{i,j}}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + \frac{k^2}{h^2}\frac{\partial^2 U_{i,j}}{\partial t^2} + \cdots$$

when applied to the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},$$

and that therefore the scheme is consistent with this differential equation only if h and k approach zero in such a way that

$$\lim_{h,k \to 0} \frac{k}{h} = 0.$$

Using the expression for the local truncation error, can you say for what partial differential equation the scheme is consistent when the mesh spacings are always chosen so that $k = ch$ for a given constant c?

(Solution on p. 28.)

SAQ 8

Is the Crank–Nicolson scheme compatible with the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}?$$

The local truncation error for the Crank–Nicolson scheme is given in SAQ 14 of *Unit 5*.

(Solution on p. 29.)

SAQ 9

Is the simple explicit scheme

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

compatible with the partial differential equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}?$$

HINT: See SAQ 15 of *Unit 5*.

(Solution on p. 29.)

We mentioned in Section 8.1 that there was a relation between stability and convergence. For the simple explicit scheme applied to the initial value problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad x \in R,\ t > 0,$$

$$U(x, 0) = f(x) \qquad x \in R,$$

we illustrated that compatibility and stability are sufficient to guarantee convergence. This condition in fact holds for other problems and forms the basis of the following result.

Lax's theorem

Suppose we are given a properly posed linear initial value problem and a linear finite-difference approximation that is compatible with it. Then the finite-difference scheme is convergent if it is stable.

This theorem applies to both partial and ordinary differential equations. It has been proved generally for initial value problems in ordinary differential equations and applies in both homogeneous and nonhomogeneous cases. For partial differential equations the theorem has been proved only in special cases, although even in cases where it has not been *proved* numerical results seem to indicate that it still applies.

In the remainder of the unit, therefore, we shall concentrate on stability. This is not to say, however, that convergence analysis is of no practical use, since from it we can obtain an estimate of the *rate* of convergence of any particular scheme.

Note that some authors give a definition of stability less restrictive than ours, for which the converse of Lax's Theorem holds, i.e. for compatible schemes convergence implies stability.

8.4 THE ANALYTICAL TREATMENT OF STABILITY

8.4.0 Introduction

Stability analysis deals with the growth of errors in a computation. In the finite-difference method we have a recurrence relation approximating a partial differential equation and we are interested in whether or not the recurrence relation induces a growth of errors as the computation proceeds. Stability analysis of recurrence relations is considered in *Unit M201 7, Introduction to Numerical Mathematics: Recurrence Relations*. For our purposes, however, the recurrence relations involve two independent arguments i and j and the analysis is somewhat different. In *Unit M201 7* we were concerned only with the growth of local rounding errors; here we have local truncation errors as well. We must determine how and when *any* local errors accumulate.

In its simplest form stability analysis consists of studying the effect of deliberately introducing errors into each stage of a computation and observing what happens as the solution to the finite-difference scheme steps forward in time. If the errors increase without bound we say that the scheme is **unstable**. On the other hand if the errors remain bounded or even die away then we call the scheme **stable**. It is often the case that a scheme is stable under certain conditions and unstable under others; we then say that the scheme is **conditionally stable**. (Note the analogy with *partial instability* for the corresponding problem in ordinary differential equations discussed in *Unit M201 21, Numerical Solution of Differential Equations*.)

8.4.1 The Matrix Method

*READ **S**: page 60, line −15,* **Analytical treatment of stability** *to page 65, line −6.*

Notes

(i) ***S**: page 61, line −5 to page 62, line 3*

We have already stated that there are two forms of error which can accumulate: local truncation errors and rounding errors. Truncation errors occur when we *discretize*, that is, replace a derivative by a difference formula (whether in the equation or in the initial or boundary conditions). Rounding errors can occur at any stage in a computation including the inaccurate storage of initial and boundary data. We must, therefore, include the effects of boundary and initial conditions in our stability analyses.

Our method is to examine the subsequent behaviour of errors at points along some time level (not necessarily the initial line) as the scheme progresses in time. For convenience we begin our calculations at $j = 0$.

(ii) ***S**: page 62, lines −9 to −3*

This sentence may be omitted as it merely restates the property of linear independence in another form.

(iii) ***S**: page 63, lines 4 to 8*

Clearly,

$$A^j \mathbf{v}_s = \lambda_s^j \mathbf{v}_s,$$

so that

$$\mathbf{e}_j = A^j\mathbf{e}_0 = A^j \sum_{s=1}^{N-1} c_s\mathbf{v}_s$$
$$= \sum_{s=1}^{N-1} c_s A^j\mathbf{v}_s$$
$$= \sum_{s=1}^{N-1} c_s \lambda_s^j \mathbf{v}_s,$$

which is Equation (3.6).

The result

$$\mathbf{e}_j = \sum_{s=1}^{N-1} c_s\lambda_s^j\mathbf{v}_s$$

shows that the only quantities involving j are powers of the eigenvalues λ_s. If we now introduce the **spectral radius** of the matrix M,

$$\rho(M) = \max_i |\lambda_i|,$$

where λ_i are the eigenvalues of M, then the previous result implies

$$\|\mathbf{e}_j\| \leqslant \sum_{s=1}^{N-1} |c_s| [\rho(A)]^j \|\mathbf{v}_s\| = [\rho(A)]^j \sum_{s=1}^{N-1} |c_s| \|\mathbf{v}_s\|.$$

If $\rho(A) < 1$ then $[\rho(A)]^j$ approaches 0 as j increases. Therefore $\mathbf{e}_j$ approaches $\mathbf{0}$, the errors die out, and the scheme is stable. If $\rho(A) = 1$ then

$$\|\mathbf{e}_j\| \leqslant \sum_{s=1}^{N-1} |c_s| \|\mathbf{v}_s\| = \text{a constant}.$$

The errors are bounded and the scheme is again stable. However, if $\rho(A) > 1$ then $\|\mathbf{e}_j\|$ increases unboundedly with increasing j and the scheme is unstable.

Thus the criterion for a stable scheme is

$$\rho(A) \leqslant 1.$$

In fact, this criterion is valid even if an eigenvector basis does not exist.

(iv) *S: page 64, lines 1 and 2*
This is shown in *Unit M201 10, Jordan Normal Form* (Section 10.2.2), when f is a polynomial function. It is easily shown that if λ is an eigenvalue of $\mathbf{B}$ then λ^{-1} is an eigenvalue of $\mathbf{B}^{-1}$.

(v) *S: page 64, lines 10 and 11*
We have

$$r \leqslant 1\Big/\left[2\sin^2\left(\frac{s\pi}{2N}\right)\right] \qquad s = 1, 2, \ldots, N-1.$$

Now, since $0 < \sin^2(s\pi/2N) < 1$, the minimum value of

$$1\Big/\left[2\sin^2\left(\frac{s\pi}{2N}\right)\right]$$

is $\frac{1}{2}$. Therefore the scheme is stable whenever $r \leqslant \frac{1}{2}$.

(vi) *S: page 65, line −6*
This equation is derived in the appendix (Section 8.7); the result quoted in *S: page 63* for T_{N-1} is a special case. The notation $i(j)k$ means

$$i, i+j, i+2j, \ldots, k.$$

General Comment

The equation discussed in the reading passage is homogeneous and has homogeneous boundary conditions. As a result its finite-difference replacement yields the homogeneous matrix equation

$$\mathbf{u}_{j+1} = A\mathbf{u}_j.$$

For problems which are nonhomogeneous in the differential equation or boundary conditions or both, the matrix equations which arise do not have the same simple form as above. This is the subject of the following SAQ.

SAQ 10

(a) Show that the matrix equations which arise in the solution of the problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad 0 < x < 1,\ t > 0,$$

$$U(x, 0) = 0 \qquad 0 \leqslant x \leqslant 1,$$

$$U(0, t) = f(t) \qquad t \geqslant 0,$$

$$U(1, t) = g(t) \qquad t \geqslant 0,$$

using the explicit scheme

$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}$$

can be written as

$$\mathbf{u}_{j+1} = A\mathbf{u}_j + \mathbf{d}_j$$

where the $\mathbf{d}_j$ are dependent on the given (known) boundary values introduced at the jth step.

(b) If an implicit scheme is used in place of the explicit scheme in (a) the matrix equations can be written as

$$B\mathbf{u}_{j+1} = A\mathbf{u}_j + \mathbf{d}_j.$$

Show that it is sufficient to investigate the eigenvalues of the matrix $B^{-1}A$ for stability if we assume that the known boundary values are given exactly.

(c) If we use a computer it is likely that the boundary values cannot be stored exactly. Determine a criterion for stability if we take into account local errors in the boundary values in $\mathbf{d}_j$.

(Solution on p. 29.)

SAQ 11

S: page 90, Exercise 2(a).

(Solution on p. 30.)

8.4.2 Two Useful Theorems

It is quite easy to develop finite-difference schemes and write them in matrix form: what is not so easy is to obtain the eigenvalues of the matrices for the stability analysis. So far we have met matrices whose eigenvalues have been known explicitly, but unfortunately this is not often the case in practice. However, we do not need to know all the eigenvalues of a matrix—we require only the eigenvalue with largest modulus (the spectral radius of the matrix). The next reading passage introduces two useful theorems which can be used in estimating spectral radii. Although it is necessary to know only the statements of the theorems a better understanding will be gained by studying their proofs (which are not too difficult).

It should be recalled that if these techniques fail to give suitable estimates for the spectral radius of a matrix we can always turn to *Unit M201 30, Numerical Solution of Eigenvalue Problems* to find numerical methods for obtaining eigenvalues; these methods are often both fast and accurate.

*READ **S**: page 65, line −5,* **Useful theorems on the bounds for eigenvalues** *to page 70, line −13.*

Notes

(i) ***S**: page 66, line −7*
It is more common to refer to Brauer's Theorem as *Gerschgorin's Circle Theorem.*

(ii) ***S**: page 68, lines 5 and 6*
Note that h_1 and h_2 are physical constants and not mesh spacings.

(iii) ***S**: page 69, line −1*
Some logic has been omitted in deriving this expression; the case $a_{s,s} = 1 - 2r$, $P_s = 2r$ leading to the condition that $r \leqslant \frac{1}{2}$ by Brauer's Theorem has been left out. The overall stability condition should be

$$r \leqslant \min\left\{\frac{1}{2 + h_1\delta x}; \frac{1}{2 + h_2\delta x}; \frac{1}{2}\right\}.$$

However, as $h_1 \geqslant 0$ and $h_2 \geqslant 0$, this condition is equivalent to the one on the last line of ***S**: page 69.*

SAQ 12

***S**: page 91, Exercise 4.*

(Solution on p. 31.)

SAQ 13

***S**: page 92, Exercise 5.*

(Solution in ***S**: page 92.*)

8.4.3 Von Neumann's Method

We have seen (*S: page 62, line 7* and SAQ 10) that the formula for propagation of any form of error introduced at every mesh point along $j = 0$ is given by (the homogeneous equation corresponding to) the finite-difference scheme. For example, consider the explicit scheme

$$u_{i,j+1} = ru_{i+1,j} - (2r - 1)u_{i,j} + ru_{i-1,j}$$

and start the computation with the vector of values $\mathbf{u}_0^*$ instead of $\mathbf{u}_0$ along $j = 0$ where

$$\mathbf{E}_0 = \mathbf{u}_0 - \mathbf{u}_0^*.$$

Then

$$E_{p,q+1} = rE_{p+1,q} - (2r - 1)E_{p,q} + rE_{p-1,q}. \tag{1}$$

[Note that we have changed the notation for the error vector from **e** to **E** in order to avoid confusion with the exponential function which we shall be using. We have also changed the notation for a mesh point from (i,j) to (p,q) so that we can use i later as the conventional symbol for $\sqrt{(-1)}$.]

Stability analysis boils down to solving recurrence relations of the type (1) subject to given initial values $E_{p,0}$ $(p = 0, 1, \ldots, N)$. Von Neumann's method, by analogy with the solution of linear partial differential equations with constant coefficients, attempts to obtain a solution to the recurrence relation by separation of variables. That is, we look for solutions in the form

$$E_{p,q} = X_p T_q \tag{2}$$

where X_p represents a function of p only and T_q a function of q only.

SAQ 14

By putting $E_{p,q} = X_p T_q$ verify that Equation (1) can be solved by the method of separation of variables. You need not solve the resulting equations for X_p and T_q.

(Solution on p. 31.)

We can obtain information about the solution by the method of separation of variables when the recurrence relation is linear and has constant coefficients. The method is outlined below.

Consider the initial values $E_{p,0}$ as values of some error function F, with domain $[0, 1]$, at $x = ph$ $(p = 0, 1, 2, \ldots, N)$. Suppose F can be expressed as a finite Fourier series. As we saw in *Unit 6, Fourier Series* it is often manipulatively more convenient to use a complex Fourier series. We therefore assume that

$$F(x) = \sum_{n=0}^{N} A_n e^{in\pi x} \qquad x \in [0, 1].$$

Since we are given the $E_{p,0}$ we have the $N + 1$ equations

$$F(ph) = \sum_{n=0}^{N} A_n e^{in\pi ph} = E_{p,0} \qquad p = 0, 1, \ldots, N, \tag{3}$$

from which the A_n may be determined. We are looking for solutions to Equation (1) which are of the type (2) and reduce to Equation (3) when $q = 0$. As the recurrence relation is linear we need only look at the behaviour of a single harmonic. Moreover, since the A_n are constants we shall only be concerned with solutions of the form

$$E_{p,q} = e^{i\beta ph} T_q,$$

where $\beta = n\pi$. If solutions of this form exist then inserting the expression for $E_{p,q}$ into the finite-difference scheme will yield an ordinary linear recurrence relation with constant coefficients for T_q, which will be satisfied by $T_q = \xi^q$ (*Unit M201 7*), for some choice of ξ. Hence our quest is for solutions of the form

$$E_{p,q} = e^{i\beta ph} \xi^q. \tag{4}$$

The remaining question is what happens to the term $E_{p,q}$ as q increases? Equation (4) indicates that $E_{p,q}$ will not increase as q increases provided that $|\xi^q| \leqslant 1$, i.e.

$$|\xi| \leqslant 1.$$

This is *von Neumann's criterion for stability.*

We apply this criterion to Equation (1) to get

$$e^{i\beta ph}\xi^{q+1} = re^{i\beta h(p+1)}\xi^q - (2r-1)e^{i\beta ph}\xi^q + re^{i\beta(p-1)h}\xi^q$$

Division by $e^{i\beta ph}\xi^q$ gives

$$\xi = re^{i\beta h} - (2r-1) + re^{-i\beta h}.$$

Recalling that

$$e^{i\beta h} + e^{-i\beta h} = 2\cos\beta h,$$

we obtain

$$\begin{aligned}\xi &= 1 + r(2\cos\beta h - 2)\\ &= 1 - 4r\sin^2\left(\frac{\beta h}{2}\right).\end{aligned}$$

Since $0 \leqslant \sin^2(\beta h/2) \leqslant 1$, von Neumann's criterion is satisfied provided $0 < r \leqslant \frac{1}{2}$.

(Compare this result with that obtained by the matrix method, ***S****: page 64.*)

READ ***S****: page 70, line −12,* **Stability by the Fourier series method** *to the end of page 72.*

SAQ 15

S*: page 93, Exercise 6(b) and (c).*

(Solution in ***S****: page 94.*)

SAQ 16 (Optional)

S*. page 94, Exercise 7.*

(Solution on p. 32.)

SAQ 17

S*: page 91, Exercise 3(b).*

(Solution in ***S****: page 91.*)

SAQ 18

Use von Neumann's method to find the stability criterion for the difference scheme

$$u_{p,q+1} - 2u_{p,q} + u_{p,q-1} = r^2c^2(u_{p+1,q} - 2u_{p,q} + u_{p-1,q})$$

applied to the hyperbolic equation

$$\frac{\partial^2 U}{\partial t^2} = c^2\frac{\partial^2 U}{\partial x^2},$$

where c is a positive constant and $r^2 = k^2/h^2$ is the mesh ratio.

Compare the result with the C.F.L. condition for this problem (SAQ 6).

(Solution on p. 33.)

The following important points should be noted concerning von Neumann's method:

(a) The method applies only to linear difference equations with constant coefficients.

(b) Von Neumann's criterion is always a necessary condition for stability. However, it is sufficient only in certain cases; in particular, this is always true of two-level schemes.

(c) The method applies, in theory, only to initial value problems with periodic initial data.

Notwithstanding the limitations stated above, von Neumann's method, being easy to apply, is used in many cases where in theory it need not apply. Often its predictions on the stability of a scheme are found to be correct, or at least very nearly so, when the scheme is used in practice.

8.5 SUMMARY

This unit has concentrated on the theory associated with finite-difference methods applied to initial value and initial-boundary value problems. We have discussed and illustrated the following points.

There are two types of error which must be considered in the analysis of finite-difference schemes. *Local truncation errors* arise from the fact that a finite-difference scheme is only an approximation to the partial differential equation. We get *local rounding errors* whenever inexact arithmetic is used to solve the finite-difference equations.

It is possible for the local errors to accumulate to such an extent that the *global error* (the difference between the computed and the true solutions) becomes so large that the computed solution bears no relation to the true solution. In general we cannot measure the local error, and hence the global error, because the local truncation errors are expressed in terms of derivatives of the true solution which we do not know, and the local rounding errors depend on the computing device used and are essentially random quantities.

For confidence in a computed solution there are two requirements of any finite-difference scheme. The first is that a reduction in the size of the mesh spacings should produce a more accurate approximation to the true solution of the partial differential equation. It follows that in the limit, as the mesh spacings are reduced to zero, the computed solution (obtained with exact arithmetic) should tend to the true solution of the differential equation. In order that this *convergence* can take place, a finite-difference scheme must also be *consistent* with the differential equation in the sense that the local truncation errors tend to zero as the mesh spacings tend to zero.

The second requirement is that any errors introduced into the numerical process should not accumulate or be amplified as the scheme progresses. We illustrated that the global error will depend on the nature of the local truncation errors, the local rounding errors and also the finite-difference scheme itself. *Stability analysis* is the study of whether local errors will or will not accumulate drastically as the step-by-step scheme progresses. We showed two commonly used methods for investigating the stability of linear finite-difference schemes; the *matrix method* and *von Neumann's method*. *Lax's Theorem* tells us that, for properly posed linear initial value problems, a linear finite-difference scheme which is both stable and consistent is also convergent. Even though Lax's Theorem tells us that we need to investigate only the stability and consistency of a scheme, convergence analyses are not redundant since they are the only means of estimating convergence rates. We therefore studied ways of establishing convergence in the case of two simple explicit methods. We also used the fact that hyperbolic equations have two real characteristics, the properties of which gave rise to the *Courant–Friedrichs–Lewy condition* for convergence of finite-difference schemes for hyperbolic equations.

8.6 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

Solution to SAQ 1

(a) We have

$$g(ih) = \frac{\partial U}{\partial t}(ih, 0) \simeq \frac{u_{i,1} - u_{i,0}}{k},$$

and so

$$u_{i,1} = f_i + kg_i,$$

where f_i represents $f(ih)$ and g_i represents $g(ik)$. Taylor's Theorem gives

$$U_{i,1} = U_{i,0} + k\frac{\partial U_{i,0}}{\partial t} + O(k^2)$$

$$= f_i + kg_i + O(k^2).$$

Thus,

$$e_{i,1} = U_{i,1} - u_{i,1} = O(k^2).$$

In fact,

$$|e_{i,1}| \leqslant \tfrac{1}{2}k^2M,$$

where M is the maximum value of $|\partial^2 U/\partial t^2|$ for $x \in R$, $t \in [0, k]$.

(b) Substituting $u_{i,j} = U_{i,j} - e_{i,j}$ into the finite-difference equation

$$u_{i,j+1} = u_{i+1,j} - u_{i,j-1} + u_{i-1,j}$$

we obtain

$$e_{i,j+1} = e_{i+1,j} - e_{i,j-1} + e_{i-1,j} + U_{i,j+1} - U_{i+1,j} + U_{i,j-1} - U_{i-1,j}. \qquad (1)$$

By definition the local truncation error $T_{i,j}$ is given by

$$T_{i,j} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} - \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$

or

$$k^2T_{i,j} = U_{i,j+1} - U_{i+1,j} + U_{i,j-1} - U_{i-1,j},$$

since $h = k$, and so Equation (1) becomes

$$e_{i,j+1} = e_{i+1,j} - e_{i,j-1} + e_{i-1,j} + k^2T_{i,j}. \qquad (2)$$

In SAQ 15 of *Unit 5* we showed that for this problem $T_{i,j} = 0$ for $p = k/h = 1$ when U is infinitely differentiable. Hence, from Equation (2), we get

$$e_{i,j+1} = e_{i+1,j} - e_{i,j-1} + e_{i-1,j}. \qquad (3)$$

(c) Replacing i by $i + 1$ and $j + 1$ by j in Equation (3), we get

$$e_{i+1,j} = e_{i+2,j-1} - e_{i+1,j-2} + e_{i,j-1},$$

and similarly, replacing i by $i - 1$ and $j + 1$ by j in Equation (3), we get

$$e_{i-1,j} = e_{i,j-1} - e_{i-1,j-2} + e_{i-2,j-1}.$$

Substitution of the last two equations into Equation (3) yields

$$e_{i,j+1} = e_{i+2,j-1} + e_{i,j-1} + e_{i-2,j-1} - e_{i+1,j-2} - e_{i-1,j-2} \qquad (4)$$

and we have expressed $e_{i,j+1}$ in terms of errors along the $j - 1$ and $j - 2$ levels. We can write down expressions for the errors along the $j - 1$ level in Equation (4), by employing the recurrence relation (3), to express $e_{i,j+1}$ in terms of errors along the $j - 2$ and $j - 3$ levels as

$$e_{i,j+1} = e_{i+3,j-2} + e_{i+1,j-2} + e_{i-1,j-2} + e_{i-3,j-2} - e_{i+2,j-3}$$
$$- e_{i,j-3} - e_{i-2,j-3}. \qquad (5)$$

We can see that Equation (3) gives us $e_{i,j+1}$ in terms of two errors along the jth level and one error along the $(j-1)$th, Equation (4) gives it in terms of three errors along the $(j-1)$th level and two along the $(j-2)$th, and Equation (5) gives it as four errors along the $(j-2)$th level and three along the $(j-3)$th. In general we can see that we can write $e_{i,j+1}$ in terms of $n+1$ errors along the $(j-n+1)$th level and n errors along the $(j-n)$th. We could prove this by induction. When $n = j$ we obtain the relation

$$e_{i,j+1} = (j+1 \text{ errors along level } 1) - (j \text{ errors along level } 0).$$

Since the same initial values along $j=0$ are used for both the differential equation and the difference equation we have $e_{i,0}=0$ for all i and, from the result of part (a), we have

$$|e_{i,1}| \leqslant \tfrac{1}{2}k^2M$$

where M is the largest value of $|\partial^2U/\partial t^2|$ in the first time interval. Hence,

$$|e_{i,j+1}| \leqslant (j+1)\tfrac{1}{2}k^2M$$

or

$$|e_{i,j}| \leqslant \tfrac{1}{2}jk^2M.$$

Now let $jk = t$, some fixed time; then $|e_{i,j}| \leqslant \tfrac{1}{2}tkM$, which approaches 0 as $k \to 0$, provided $\partial^2U/\partial t^2$ is bounded in the first time interval.

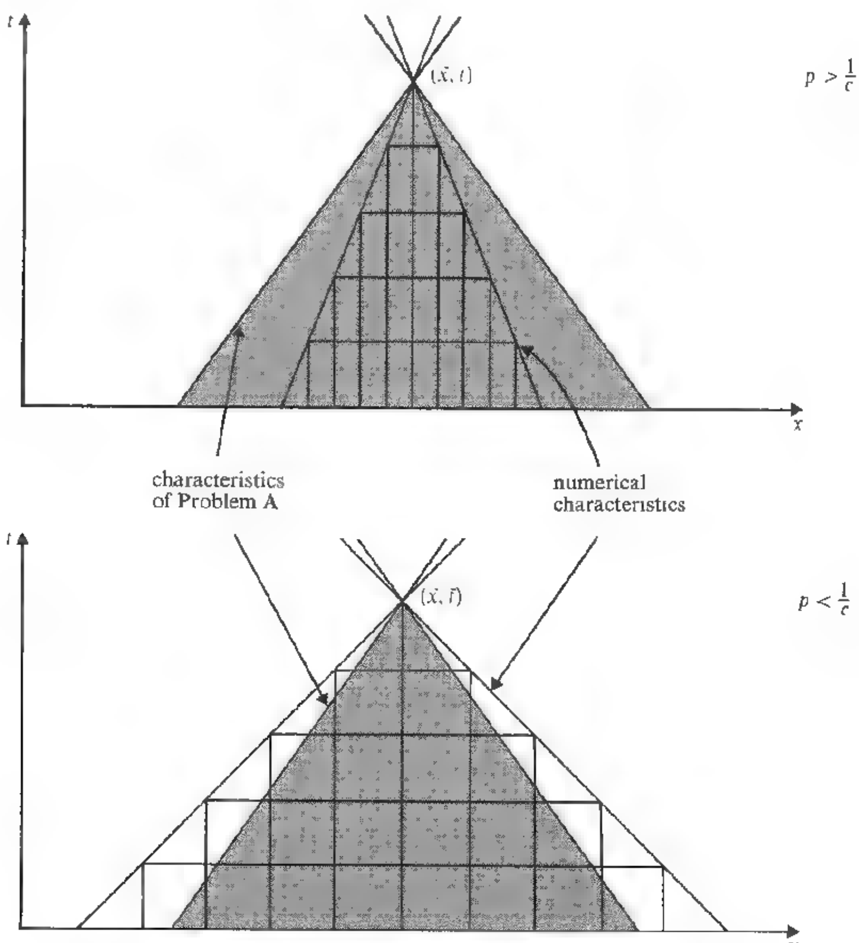
Therefore the scheme is convergent.

Solution to SAQ 2

(a) The solution to Problem A at $(\bar{x}, \bar{t})$ depends on the data on that part of the initial line included in the domain of dependence of (x, t).

(b) The solution to Scheme B at $(x, \bar{t})$ depends on the data at those mesh points at level 0 which are included in the numerical domain of dependence of $(x, \bar{t})$.

Solution to SAQ 3



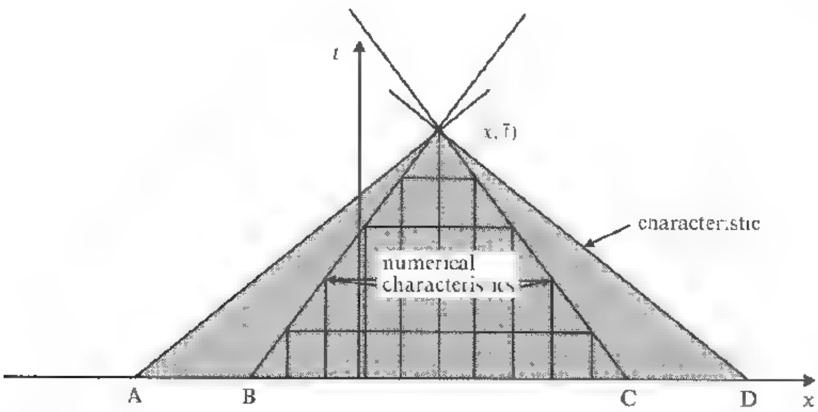
The domains of dependence (for Problem A) are shaded in the diagrams; the hatched areas are the numerical domains of dependence (for Scheme B).

When $p > 1/c$ the numerical domain of dependence of the point $(\bar{x}, \bar{t})$ lies within the domain of dependence of the same point. When $p < 1/c$ the numerical domain of dependence lies outside the domain of dependence. The numerical domain of dependence varies as the mesh ratio p is altered, and in general the disposition of the numerical domain of dependence of a point relative to the domain of dependence of the same point depends on whether $p < 1/c$ or $p > 1/c$.

The domains of dependence of $(\bar{x}, \bar{t})$ for Problem A and Scheme B coincide when $p = 1/c$.

Solution to SAQ 4

In the diagram the initial line segment BC corresponds to (i) and the initial line segments AB and CD correspond to (ii).



Any changes to the initial data along BC will cause changes in the analytical solution *and* in the numerical solution at the point $(\bar{x}, \bar{t})$. However, changes along AB or CD will cause a change in the analytical solution but *not* in the numerical solution. Therefore, no matter how good an approximation the numerical scheme is to the hyperbolic equation, the solution of the numerical scheme cannot respond to changes in the initial data in the way that the analytical solution does.

We have shown the characteristics of the hyperbolic equation as straight lines since, in Problem A, c is a constant. The argument of the solution to this SAQ still holds when the characteristics of the given hyperbolic equation are curved.

Solution to SAQ 5

By the result of SAQ 4 the mesh ratio for a finite-difference scheme applied to a hyperbolic equation should be chosen so that the numerical domain of dependence includes the domain of dependence.

For Problem A, since the characteristics are straight lines, the mesh ratio should be chosen so that

$$p \leqslant \frac{1}{c}.$$

Solution to SAQ 6

The characteristics of

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2}$$

through the point $(\bar{x}, \bar{t})$ are given by

$$t - \bar{t} = \pm \frac{1}{c}(x - \bar{x}). \tag{1}$$

The numerical characteristics of the scheme

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = p^2c^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

through $(x, \bar{t})$ are given by

$$t - \bar{t} = \pm p(x - \bar{x}). \tag{2}$$

The C.F.L. condition requires that the numerical domain of dependence of $(\bar{x}, \bar{t})$ should include the domain of dependence of the same point. This implies that the slopes of (2) be less than or equal to the slopes of (1) in magnitude. That is

$$p \leqslant \frac{1}{c}.$$

Solution to SAQ 7

Using Taylor's Theorem, we find that

$$U_{i,j+1} - U_{i,j-1} = 2k\frac{\partial U_{i,j}}{\partial t} + \frac{k^3}{3}\frac{\partial^3 U_{i,j}}{\partial t^3} + O(k^5),$$

$$U_{i,j+1} + U_{i,j-1} = 2U_{i,j} + k^2\frac{\partial^2 U_{i,j}}{\partial t^2} + \frac{k^4}{12}\frac{\partial^4 U_{i,j}}{\partial t^4} + O(k^6),$$

and

$$U_{i+1,j} + U_{i-1,j} = 2U_{i,j} + h^2\frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + O(h^6).$$

Therefore, the local truncation error is given by

$$\begin{aligned}
T_{i,j} &= \frac{U_{i,j+1} - U_{i,j-1}}{2k} - \frac{U_{i+1,j} - U_{i,j+1} - U_{i,j-1} + U_{i-1,j}}{h^2} \\
&= \frac{\partial U_{i,j}}{\partial t} + \frac{k^2}{6}\frac{\partial^3 U_{i,j}}{\partial t^3} + O(k^4) \\
&\quad - \left(\frac{\partial^2 U_{i,j}}{\partial x^2} - \frac{k^2}{h^2}\frac{\partial^2 U_{i,j}}{\partial t^2} + \frac{h^2}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} - \frac{k^4}{12h^2}\frac{\partial^4 U_{i,j}}{\partial t^4} + O(h^4) + O(k^6/h^2)\right) \\
&= \frac{\partial U_{i,j}}{\partial t} - \frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{k^2}{6}\frac{\partial^3 U_{i,j}}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + \frac{k^2}{h^2}\frac{\partial^2 U_{i,j}}{\partial t^2} \\
&\qquad + O(k^4) + O(h^4) + O(k^4/h^2).
\end{aligned}$$

Then, for the differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},$$

the local truncation error is

$$T_{i,j} = \frac{k^2}{6}\frac{\partial^3 U_{i,j}}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + \frac{k^2}{h^2}\frac{\partial^2 U_{i,j}}{\partial t^2} + O(k^4) + O(h^4) + O(k^4/h^2).$$

We can see that the expression for the local truncation error contains a term involving k^2/h^2, which does not approach zero as $h, k \to 0$ unless

$$\lim_{h,k \to 0} \frac{k}{h} = 0.$$

If $k/h = c$ then the equation for the local truncation error becomes

$$\begin{aligned}
T_{i,j} &= \frac{\partial U_{i,j}}{\partial t} - \frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{k^2}{6}\frac{\partial^3 U_{i,j}}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + c^2\frac{\partial^2 U_{i,j}}{\partial x^2} \\
&\qquad + O(k^4) + O(h^4) + O(c^2k^2),
\end{aligned}$$

which approaches 0 as $h, k \to 0$ if U satisfies the differential equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + c^2 \frac{\partial^2 U}{\partial t^2} = 0.$$

which is hyperbolic! Thus the scheme is consistent with this partial differential equation provided the mesh spacing is always chosen such that $k = ch$, where c is constant.

Solution to SAQ 8

From SAQ 14 of *Unit 5* the local truncation error of the Crank–Nicolson implicit method applied to

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

is given by

$$T_{i,j} = O(k^2) + O(h^2),$$

from which we see immediately that $T_{i,j}$ approaches 0 as $k, h \to 0$ and the scheme is compatible with the differential equation.

Solution to SAQ 9

From SAQ 15 of *Unit 5* the local truncation error of the simple explicit scheme

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

applied to the differential equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}$$

is given by

$$T_{i,j} = 0,$$

and the scheme is compatible with the differential equation.

Solution to SAQ 10

(a) The jth matrix equation is

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ & \cdot & \cdot & \cdot & \\ & & & r & (1-2r) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-1,j} \end{bmatrix} + \begin{bmatrix} rf_j \\ 0 \\ \vdots \\ rg_j \end{bmatrix}.$$

which can be written as

$$\mathbf{u}_{j+1} = A\mathbf{u}_j + \mathbf{d}_j.$$

(b) The Crank–Nicolson formula is given by the equation on the last line of *S: page 64*. The inclusion of boundary values means that we have to add the column vector $(rf_j, 0, 0, \ldots, rg_j)$ to the right-hand side of this equation which now takes the form

$$B\mathbf{u}_{j+1} = A\mathbf{u}_j + \mathbf{d}_j.$$

(The same form would be obtained if we used some other implicit method.) This last equation can be written as

$$\mathbf{u}_{j+1} = B^{-1}A\mathbf{u}_j + B^{-1}\mathbf{d}_j$$

provided $\det B \neq 0$ (which is the same as saying that it is possible to solve the given matrix equation).

If we now introduce errors at every point along $j = 0$ and begin the computation with the vector $\mathbf{u}_0^*$ instead of $\mathbf{u}_0$ we see that for $j = 0, 1, 2, \ldots$

$$\mathbf{u}_{j+1}^* = B^{-1}A\mathbf{u}_j^* + B^{-1}\mathbf{d}_j, \tag{1}$$

provided that there are no errors in the $\mathbf{d}_j$.

Putting

$$\mathbf{e}_j = \mathbf{u}_j - \mathbf{u}_j^*,$$

we obtain

$$\mathbf{e}_{j+1} = B^{-1}A\mathbf{e}_j$$

which implies that

$$\mathbf{e}_j = (B^{-1}A)^j\mathbf{e}_0$$

which does not involve any boundary values, and we need only consider the eigenvalues of $B^{-1}A$.

(c) If we do introduce errors into the boundary values we have to write Equation (1) as

$$\mathbf{u}_{j+1}^* = B^{-1}A\mathbf{u}_j^* + B^{-1}\mathbf{d}_j^*$$

where the $\mathbf{d}_j^*$ are the boundary values $\mathbf{d}_j$ with errors introduced. Putting $\mathbf{e}_j = \mathbf{u}_j - \mathbf{u}_j^*$ once more, we obtain

$$\mathbf{e}_{j+1} = B^{-1}A\mathbf{e}_j + B^{-1}(\mathbf{d}_j - \mathbf{d}_j^*).$$

Putting $\mathbf{b}_j$ for $B^{-1}(\mathbf{d}_j - \mathbf{d}_j^*)$, we see that

$$\mathbf{e}_j = (B^{-1}A)^j\mathbf{e}_0 + \sum_{k=0}^{j-1} (B^{-1}A)^{j-1-k}\mathbf{b}_k.$$

Again, the error vector is bounded and the scheme is stable provided the spectral radius of $B^{-1}A$ is less than or equal to one.

Solution to SAQ 11

The difference equation may be written as

$$(1 + 2ra)u_{i,j+1} - ra(u_{i-1,j+1} + u_{i+1,j+1}) = u_{i,j}$$

which, in matrix form, is

$$\begin{bmatrix} 1+2ra & ra & & & \\ -ra & 1+2ra & -ra & & \\ & \cdot & \cdot & \cdot & \\ & & -ra & 1+2ra & -ra \\ & & & -ra & 1+2ra \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \cdot \\ u_{N-2,j+1} \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \cdot \\ u_{N-2,j} \\ u_{N-1,j} \end{bmatrix} + \begin{bmatrix} d_{1,j} \\ 0 \\ \cdot \\ 0 \\ d_{N-1,j} \end{bmatrix}$$

where the $d_{i,j}$ are obtained from the known boundary values. We can write this equation as

$$(I - raT_{N-1})\mathbf{u}_{j+1} = \mathbf{u}_j + \mathbf{d}_j$$

where T_{N-1} is a square matrix of order $N - 1$ given by

$$T_{N-1} = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$

Thus

$$\mathbf{u}_{j+1} = (I - raT_{N-1})^{-1}\mathbf{u}_j + (I - raT_{N-1})^{-1}\mathbf{d}_j$$

provided $\det(I - raT_{N-1}) \neq 0$. The eigenvalues of $(I - raT_{N-1})^{-1}$ are (using the eigenvalues of T_{N-1} given in *S: page 63*)

$$\frac{1}{1 + 4ra\sin^2\left(\dfrac{s\pi}{2N}\right)} \qquad s = 1, 2, \ldots, N-1,$$

and these are strictly less than 1 in modulus for all positive values of r, so that the difference scheme is unconditionally stable.

Solution to SAQ 12

The scheme of part (a) yields

$$\mathbf{u}_{j+1} = A\mathbf{u}_j$$

where

$$A = \begin{bmatrix} 1-2r & r & & & & \\ r & 1-2r & r & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & r & 1-2r & r \\ & & & & r & 1-2r \end{bmatrix}$$

and the solution follows by the method of *S: page 92*.

Solution to SAQ 13

See the solution in *S: page 92*.

Solution to SAQ 14

Putting $E_{p,q} = X_pT_q$ in

$$E_{p,q+1} = rE_{p+1,q} - (2r-1)E_{p,q} + rE_{p-1,q},$$

we obtain

$$X_pT_{q+1} = rX_{p+1}T_q - (2r-1)X_pT_q + rX_{p-1}T_q.$$

Dividing by X_pT_q, we obtain

$$\frac{T_{q+1}}{T_q} = \frac{rX_{p+1} - (2r-1)X_p + rX_{p-1}}{X_p}$$

and, since the left-hand side of the equation is a function of q only and the right-hand side is a function of p only, the equation is separable as

$$T_{q+1} = \xi T_q$$

$$rX_{p+1} - (2r-1)X_p + rX_{p-1} = \xi X_p$$

where ξ is a constant. (In principle we can solve these two recurrence relations by the methods of *Unit M201 7*.)

Solution to SAQ 15

See the solution in *S: page 94*.

Solution to SAQ 16

Since the error function E satisfies the original difference equation (with homogeneous boundary conditions), substitution of $E_{p,q} = e^{i\beta ph}\xi^q$ gives

$$e^{i\beta ph}\xi^{q+1} - e^{i\beta ph}\xi^q = \frac{ka}{h^2}\{e^{i\beta(p-1)h}\xi^q - 2e^{i\beta ph}\xi^q + e^{i\beta(p+1)h}\xi^q\} + kbe^{i\beta ph}\xi^q.$$

Division by $e^{i\beta ph}\xi^q$ gives

$$\xi - 1 = \frac{ka}{h^2}(e^{-i\beta h} - 2 + e^{i\beta h}) + kb.$$

Hence,

$$\xi = kb + 1 - \frac{4ka}{h^2}\sin^2\left(\frac{\beta h}{2}\right).$$

For stability we require $|\xi| \leqslant 1$ and therefore we need

$$-1 \leqslant kb + 1 - \frac{4ka}{h^2}\sin^2\left(\frac{\beta h}{2}\right) \leqslant 1.$$

Recalling that $\beta = n\pi$ and $Nh = 1$ we require the following.

(i) $$kb + 1 - \frac{4ka}{h^2}\sin^2\frac{n\pi}{2N} \leqslant 1 \qquad 1 \leqslant n \leqslant N-1,$$

since the boundary values are known. This is equivalent to

$$b \leqslant \frac{4a}{h^2}\sin^2\frac{n\pi}{2N},$$

which we may replace by

$$h^2 \leqslant \frac{4a}{b}\sin^2\frac{\pi}{2N},$$

since

$$\sin^2\frac{\pi}{2N} \leqslant \sin^2\frac{n\pi}{2N} \qquad 1 \leqslant n \leqslant N-1.$$

Hence,

$$h \leqslant 2\sqrt{\frac{a}{b}}\sin\frac{\pi}{2N},$$

as required.

(ii) $$-1 \leqslant kb + 1 - \frac{4ka}{h^2}\sin^2\frac{n\pi}{2N} \qquad 1 \leqslant n \leqslant N-1$$

is equivalent to

$$4ka\sin^2\frac{n\pi}{2N} \leqslant (kb+2)h^2,$$

i.e.,

$$2a\sin^2\frac{n\pi}{2N} \leqslant \frac{h^2}{k} + \tfrac{1}{2}h^2 b,$$

so that

$$\frac{h^2}{k} \geqslant 2a\sin^2\frac{n\pi}{2N} - \tfrac{1}{2}h^2 b, \quad \text{or}$$

$$\frac{k}{h^2} \leqslant \frac{1}{2a\sin^2\frac{n\pi}{2N} - \frac{1}{2}h^2 b} \qquad 1 \leqslant n \leqslant N-1.$$

Since

$$1 > \sin^2\frac{n\pi}{2N} \qquad 1 \leqslant n \leqslant N-1,$$

we have

$$\frac{k}{h^2} \leqslant \frac{1}{2a - \frac{1}{2}h^2 b},$$

as required.

Solution to SAQ 17

See the solution in *S: page 91*.

Solution to SAQ 18

Putting $E_{p,q} = e^{i\beta ph}\xi^q$ in

$$E_{p,q+1} - 2E_{p,q} + E_{p,q-1} = r^2c^2(E_{p+1,q} - 2E_{p,q} + E_{p-1,q}),$$

we obtain

$$e^{i\beta ph}\xi^{q+1} - 2e^{i\beta ph}\xi^q + e^{i\beta ph}\xi^{q-1} = r^2c^2(e^{i\beta(p+1)h}\xi^q - 2e^{i\beta ph}\xi^q + e^{i\beta(p-1)h}\xi^q).$$

Hence,

$$\xi - 2 + \xi^{-1} = r^2c^2(e^{i\beta h} - 2 + e^{-i\beta h}),$$

i.e.,

$$\xi - 2 + \xi^{-1} = -r^2c^2 4\sin^2\left(\frac{\beta h}{2}\right).$$

Thus

$$\xi^2 - 2\left(1 - 2r^2c^2\sin^2\left(\frac{\beta h}{2}\right)\right)\xi + 1 = 0,$$

which gives

$$\xi_1, \xi_2 = 1 - 2r^2c^2\sin^2\left(\frac{\beta h}{2}\right) \pm 2rc\sin\left(\frac{\beta h}{2}\right)\left[r^2c^2\sin^2\left(\frac{\beta h}{2}\right) - 1\right]^{\frac{1}{2}}.$$

Now, for stability we require

$$|\xi_l| \leqslant 1 \qquad l = 1, 2.$$

If $r^2c^2\sin^2(\beta h/2) \leqslant 1$, we have

$$\xi_1, \xi_2 = 1 - 2r^2c^2\sin^2\left(\frac{\beta h}{2}\right) \pm 2irc\sin\left(\frac{\beta h}{2}\right)\left[1 - r^2c^2\sin^2\left(\frac{\beta h}{2}\right)\right]^{\frac{1}{2}}$$

and $|\xi_l| = 1 \qquad l = 1, 2.$

On the other hand, if $r^2c^2\sin^2(\beta h/2) > 1$ we get $|\xi_2| > 1$.

Therefore

$$|\xi_l| \leqslant 1 \qquad l = 1, 2,$$

holds only when $r^2c^2\sin^2(\beta h/2) \leqslant 1$, and so is true for all β if $r^2c^2 \leqslant 1$. Thus, von Neumann's condition for stability is $k/h \leqslant 1/c$. From SAQ 6 the C.F.L. condition for convergence is also $k/h \leqslant 1/c$.

8.7 APPENDIX

The Eigenvalues of a Common Tridiagonal Matrix

The following is a derivation of the formula given in *S: page 65* for the eigenvalues of the tridiagonal matrix of order N given by

$$\begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix},$$

where b and c are real and have the same sign.

By the definition of an eigenvalue we have the matrix equation

$$\begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \\ \\ v_{N-1} \\ v_N \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_{N-1} \\ v_N \end{bmatrix},$$

where $(v_1, \ldots, v_N)$ is the eigenvector associated with the eigenvalue λ. We see that this matrix equation is equivalent to the set of equations

$$av_1 + bv_2 = \lambda v_1,$$

$$cv_{r-1} + av_r + bv_{r+1} = \lambda v_r \qquad r = 2, 3, \ldots, N-1,$$

$$cv_{N-1} + av_N = \lambda v_N.$$

If we define $v_0 = v_{N+1} = 0$, the above equations can be written concisely as

$$cv_{r-1} + av_r + bv_{r+1} = \lambda v_r \qquad r = 1, 2, \ldots, N, \tag{1}$$

which is a second-order recurrence relation subject to the boundary conditions

$$v_0 = v_{N+1} = 0. \tag{2}$$

The solution of (1) is given by (see *Unit M201 7*)

$$v_r = d_1 p_1^r + d_2 p_2^r$$

where p_1 and p_2 are the roots of the quadratic equation

$$bp^2 + (a - \lambda)p + c = 0$$

and d_1 and d_2 are arbitrary constants. Now, using a trick from *Unit M201 25* (Section 25.4), we make the substitution $\lambda - a = 2(bc)^{\frac{1}{2}} \cos\theta$. So the quadratic equation becomes

$$p^2 - 2\left(\frac{c}{b}\right)^{\frac{1}{2}} \cos\theta \, p + \frac{c}{b} = 0,$$

and the roots are

$$p_1, p_2 = \left(\frac{c}{b}\right)^{\frac{1}{2}} (\cos\theta \pm i \sin\theta).$$

The general solution to the recurrence relation (1) is therefore, by De Moivre's Theorem,

$$v_r = \left(\frac{c}{b}\right)^{r/2}(d_3 \cos r\theta + d_4 \sin r\theta)$$

where $d_3 = d_1 + d_2$ and $d_4 = i(d_1 - d_2)$.

The boundary conditions (2) yield

$$d_3 = 0$$

and

$$\left(\frac{c}{b}\right)^{(N+1)/2} d_4 \sin(N + 1)\theta = 0,$$

for which

$$\theta = \frac{s\pi}{N + 1} \qquad s = 1, 2, \ldots, N.$$

Thus the sth eigenvector has its rth component given by

$$v_{s,r} = \left(\frac{c}{b}\right)^{r/2} \sin\frac{rs\pi}{N + 1},$$

that is,

$$\mathbf{v}_s = \left\{\left(\frac{c}{b}\right)^{\frac{1}{2}} \sin\frac{s\pi}{N + 1}, \left(\frac{c}{b}\right) \sin\frac{2s\pi}{N + 1}, \ldots, \left(\frac{c}{b}\right)^{N/2} \sin\frac{Ns\pi}{N + 1}\right\}$$

with corresponding eigenvalue λ_s given by

$$\lambda_s = a + 2(bc)^{\frac{1}{2}} \cos\frac{s\pi}{N + 1}.$$

Unit 9 Green's Functions I: Ordinary Differential Equations

Contents

Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).

H. F. Weinberger, *A First Course in Partial Differential Equations* (Blaisdell, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as S and W respectively.

Unit 9 is based on W: Chapter V, Sections 27 and 28.

Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics.* References to Open University courses in mathematics take the form:

Unit M100 13, Integration II for the Mathematics Foundation Course,

Unit M201 23, The Wave Equation for the Linear Mathematics Course.

Bibliography

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

This book contains an excellent account of Green's functions for boundary value problems in Chapter 12 (Sections 12-9 to 12-11). The approach is from a slightly different viewpoint from that adopted in this unit, in a style which will be familiar to students who have taken M201.

9.0 INTRODUCTION

The present unit treats ordinary differential equations, subject to initial conditions and boundary conditions, from a fresh point of view. Firstly we summarize some of the principal properties already known to you.

The general *linear equation of the second order* is of the form

$$a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)u = f(x), \tag{1}$$

and the *associated homogeneous equation* is

$$a_2(x)\frac{d^2v}{dx^2} + a_1(x)\frac{dv}{dx} + a_0(x)v = 0. \tag{2}$$

The solution space of Equation (2) has dimension 2; that is to say, any three solutions of (2) are linearly dependent, and there exist two solutions which are linearly independent. If we call a given linearly independent pair of solutions v_1 and v_2, then for any constants c_1 and c_2,

$$c_1v_1 + c_2v_2$$

is a solution of (2). Moreover, every solution of (2) is of this form (*Unit M201 9, Differential Equations II*).

Let u_p be any solution of (1). Then (1) has the solution set

$$\{u : u = u_p + c_1v_1 + c_2v_2;\ c_1, c_2 \in R\}. \tag{3}$$

Practical methods are needed for obtaining u_p, v_1 and v_2. When a_2, a_1, a_0 are *constants*, v_1 and v_2 are easy to obtain (*Unit M201 9*), but otherwise, a special investigation is normally needed. To find a particular solution u_p we may employ intelligent guesswork for simple cases, or use a systematic method such as Variation of Parameters (*Unit M201 11, Differential Equations III*).

One solution in the set (3) may be singled out if *initial conditions* are supplied, of the form

$$u(\alpha) = u_0, \quad u'(\alpha) = u_1.$$

This requirement says that the solution curve sought must pass through the point $x = \alpha$, $u = u_0$, and have slope equal to u_1 at $x = \alpha$. The Existence and Uniqueness Theorem (*Unit M201 33, Existence and Uniqueness Theorem for Differential Equations*) states that, under quite general conditions on a_2, a_1, a_0 and f (which are satisfied for all the specific equations considered in this unit), there is one, and only one, solution to Equation (1) which satisfies the initial conditions.

In this unit we shall consider principally another class of problems, using the above theory as a starting point. These are *boundary value problems*—to find a function u satisfying

$$a_2u'' + a_1u' + a_0u = f \qquad \text{in } (\alpha, \beta),$$

with

$$u(\alpha) = a, \quad u(\beta) = b.$$

We have met such problems in *Unit M201 25, Boundary Value Problems*. Here we require, if possible, a solution curve passing through the points (α, a), (β, b).

The Existence Theorem does not say anything directly about these problems; there may be no solution, there may be one and only one, or there may be an infinite number. Simple examples of each possibility are the following:

(i) $u'' + u = 0 \qquad \text{in } (0, \pi)$,

$$u(0) = 0, \quad u(\pi) = 1.$$

This has no solution.

(ii) $u'' + u = 1 \qquad$ in $(0, \pi)$,

$\qquad u(0) = 0, \quad u(\pi) = 0.$

This has the unique solution $u(x) = 1 - \sin x - \cos x$.

(iii) $u'' + u = 1 \qquad$ in $(0, \pi)$,

$\qquad u(0) = 1, \quad u(\pi) = 1.$

This has an infinite number of solutions, $u(x) = 1 + C \sin x$, for any value of C.

The occurrence of these possibilities does not really depend on delicate properties of f and the coefficients; the boundary value problem has completely new features differing from those of the initial value problem. The method of *Green's functions* which is discussed in this unit enables these possibilities to be investigated, and allows the solution to be written down when it exists. The method applies also to differential equations in domains of dimension greater than one (that is to say, partial differential equations) and *Unit 10, Green's Functions II* will deal with this topic.

Our method of attack is as follows. We begin by constructing a *formula* for the solution to the nonhomogeneous initial value problem, in terms of an *influence function*. We then use this result to give us some leverage on the nonhomogeneous boundary value problem, for which we obtain a formula in terms of a *Green's function*. The Green's function is essentially a continuous, but not smooth, "solution" to the associated homogeneous problem. The basic Green's function solution is refined (in Section 9.2.2) to deal with the most general type of (unmixed) boundary conditions, and a physical interpretation is given in Section 9.2.3.

9.1 INITIAL VALUE PROBLEMS

READ W: Section 27, pages 117 to 120.

Notes

(i) *W: page 117, first paragraph*
The transformation leading to Equation (27.1) is obtained as follows. When the term $(pu')'$ is expanded, the equation becomes

$$pu'' + p'u' + qu = f \qquad \text{in } (\alpha, \beta),$$

and we require this equation to be equivalent to the original equation

$$au'' + bu' + cu = F \qquad \text{in } (\alpha, \beta),$$

that is to say, to have the same solution set.

Assume that a and p have nonzero values throughout (α, β); then we require the equations

$$u'' + \frac{p'}{p}u' + \frac{q}{p}u = \frac{f}{p}$$

and

$$u'' + \frac{b}{a}u' + \frac{c}{a}u = \frac{F}{a}$$

to be equivalent on (α, β). This is achieved if the coefficients are the same, i.e. if

$$\frac{p'}{p} = \frac{b}{a}, \quad \frac{q}{p} = \frac{c}{a}, \quad \frac{f}{p} = \frac{F}{a}. \tag{1}$$

The first of these is a differential equation, with a solution

$$p(x) = \exp\left(\int^x \frac{b(\xi)}{a(\xi)}\,d\xi\right),$$

where

$$\int \frac{b}{a}$$

is any indefinite integral of b/a. (The one chosen in W is

$$\int_\alpha^x \frac{b(\xi)}{a(\xi)}\,d\xi.$$

In the footnote it is explained that when $\lim_{\xi \to \alpha} a(\xi)$ is zero it might be necessary to choose another lower limit—in other words another indefinite integral.)

The functions q and f are now obtained from Equations (1).

(ii) *W: page 118, line −5*
We have $K = p(x)W[v_1, v_2](x)$ for all $x \in (\alpha, \beta)$ where W denotes the Wronskian (*Unit M201 9*). Clearly $p(x)$ is nonzero throughout (α, β), and the Wronskian is nonzero since v_1 and v_2 are linearly independent. Hence, $K \neq 0$.

(iii) *W: page 119, line 6*
In particular, R is independent of any particular choice of v_1 and v_2.

(iv) *W: page 119, lines −11, −10*
The function

$$u_h : x \longmapsto \int_\alpha^x R(x, \xi) f(\xi)\,d\xi$$

is the particular solution of the equation

$$(pu')' + qu = f$$

which satisfies the (homogeneous) initial conditions $u_h(\alpha) = u_h'(\alpha) = 0$. The solution set of the differential equation is given by

$$\{u : u = u_h + c_1 v_1 + c_2 v_2 : c_1, c_2 \in R\},$$

where v_1 and v_2 are any two linearly independent solutions of the associated homogeneous equation

$$(pv')' + qv = 0.$$

The system

$$(pu')' + qu = f \qquad \text{in } (\alpha, \beta),$$

$$u(\alpha) = a, \quad u'(\alpha) = b$$

has a unique solution for every pair of constants a, b (*Unit M201 33*), and this solution can be expressed in the form $u_h + c_1 v_1 + c_2 v_2$ for some pair of constants c_1, c_2.

We find the constants from the following equations:

$$u(\alpha) = u_h(\alpha) + c_1 v_1(\alpha) + c_2 v_2(\alpha),$$

i.e.

$$a = c_1 v_1(\alpha) + c_2 v_2(\alpha), \qquad (*)$$

and

$$u'(\alpha) = u_h'(\alpha) + c_1 v_1'(\alpha) + c_2 v_2'(\alpha),$$

i.e.

$$b = c_1 v_1'(\alpha) + c_2 v_2'(\alpha). \qquad (*)$$

General Comment

The derivation of the influence function given here is similar to the method of Variation of Parameters described in *Unit M201 11*. In the notation of that unit, the equation on *W: page 118, line 4* is essentially of the form

$$y_p(x) = y_1(x)c_1(x) + y_2(x)c_2(x)$$

where foreknowledge has provided the author with the correct form of the functions c_1 and c_2. The following table illustrates the relationship of the notation used in *W* and that in *Unit M201 11*.

M201	*W*
$y'' + a_1 y' + a_0 y = h$	$(pv')' + qv = f$
a_1, a_0, h	$p'/p, q/p, f/p$
y_1, y_2	v_1, v_2
y_p	w/K
x_0	α
Green's function $K(x, t)$ for an initial value problem.	influence function $R(x, \xi)$

SAQ 1

Reduce the following equation to self-adjoint form:

$$(1 + x^2)u''(x) + 2xu'(x) + \lambda u(x) = 0 \qquad -1 < x < 1.$$

(Solution on p. 24.)

SAQ 2

Reduce the following equation to self-adjoint form:

$$u''(x) + \frac{1}{x}u'(x) - \frac{n^2}{x^2}u(x) = F(x) \qquad 0 < x < 1.$$

(Solution on p. 24.)

SAQ 3

Find the influence function for the equation

$$u''(x) - u(x) = e^x \qquad x > 0,$$

and obtain the solution satisfying

$$u(0) = 1, \quad u'(0) = 0.$$

(Solution on p. 24.)

SAQ 4

Reduce the equation

$$x^2u''(x) - 2xu'(x) + 2u(x) = x^4 \qquad x > 1$$

to self-adjoint form.

Determine two linearly independent solutions v_1 and v_2 of the associated homogeneous equation, and confirm that $p(x)[v_1'(x)v_2(x) - v_2'(x)v_1(x)]$ is constant.

Obtain the influence function R, and use it to solve the system

$$x^2u''(x) - 2xu'(x) + 2u(x) = x^4 \qquad x > 1,$$
$$u(1) = 1, \quad u'(1) = 0.$$

HINT: To solve the homogeneous equation, look for a solution of the form $x \longmapsto x^n$.

(Solution on p. 25.)

9.2 BOUNDARY VALUE PROBLEMS

9.2.1 Homogeneous Boundary Conditions

In Section 9.1 we saw how the initial value problem:

$$(pu')' + qu = f \qquad \text{in } (\alpha, \beta),$$
$$u(\alpha) = 0, \quad u'(\alpha) = 0, \tag{1}$$

can be solved by constructing the influence function $R(x, \xi)$. This requires only a knowledge of any two linearly independent solutions, v_1 and v_2, of the associated homogeneous equation $(pv')' + qv = 0$. The solution of (1) can then be written in the form

$$u(x) = \int_\alpha^x R(x, \xi) f(\xi) d\xi. \tag{2}$$

If the initial conditions in (1) are nonhomogeneous, so that $u(\alpha) = a$, $u'(\alpha) = b$ where a and b are not both zero, the problem can be solved by adding suitable multiples of v_1 and v_2 to (2), as in *W: page 119*.

In this course we are concerned as much with boundary value problems, where a condition is imposed at each end of an interval, as with initial value problems. A typical example is

$$(pu')' + qu = f \qquad \text{in } (\alpha, \beta),$$
$$u(\alpha) = 0, \quad u(\beta) = 0. \tag{3}$$

This system has *homogeneous* boundary conditions, but the equation is nonhomogeneous. We shall find again that the solution can be represented as an integral, this time in the form

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi,$$

where the function G turns out to be constructed from solutions of the homogeneous equation, as the influence function was.

This method provides a useful way of displaying the solution, which, as we shall see later on, has a strong physical motivation. It also enables the question of the existence and uniqueness of solutions of the system (3), and of the analogous system with non-homogeneous boundary conditions, to be resolved. Remember that the existence and uniqueness of solutions of ordinary differential equations has so far been considered for initial value problems only.

READ W: page 120, Section 28 to page 123, line 2 (the end of the example).

Notes

(i) *W: page 121, line 11*
The case $D = 0$ is discussed later.

(ii) *W: page 121, line −8*
You will be asked to fill in this manipulation in SAQ 5.

(iii) *W: page 122, lines 4 to 6*
It is very important to grasp the meaning of the statement "G is symmetric". Merely to exchange x and ξ in the algebraic expressions in Equation (28.2) (*W: page 121*) does not return $G(x, \xi)$ unless they are also exchanged in the inequalities. The following notation is helpful. Let

$$G^+(x, \xi) = \frac{1}{KD}[v_1(\xi)v_2(\alpha) - v_2(\xi)v_1(\alpha)][v_1(x)v_2(\beta) - v_2(x)v_1(\beta)]$$

and

$$G^-(x, \xi) = \frac{1}{KD}[v_1(x)v_2(\alpha) - v_2(x)v_1(\alpha)][v_1(\xi)v_2(\beta) - v_2(\xi)v_1(\beta)]$$

for $(x, \xi) \in [\alpha, \beta] \times [\alpha, \beta]$. Then

$$G(x, \xi) = \begin{cases} G^+(x, \xi) & \xi \leqslant x, \\ G^-(x, \xi) & x \leqslant \xi. \end{cases}$$

If we exchange ξ and x in the *whole* expression above (so that $\xi \leqslant x$ becomes $x \leqslant \xi$ and vice versa), we get

$$G(\xi, x) = \begin{cases} G^+(\xi, x) & x \leqslant \xi, \\ G^-(\xi, x) & \xi \leqslant x. \end{cases}$$

But

$$G^+(\xi, x) = G^-(x, \xi) \quad \text{and} \quad G^-(\xi, x) = G^+(x, \xi),$$

so that

$$G(\xi, x) = \begin{cases} G^-(x, \xi) & x \leqslant \xi, \\ G^+(x, \xi) & \xi \leqslant x, \end{cases}$$

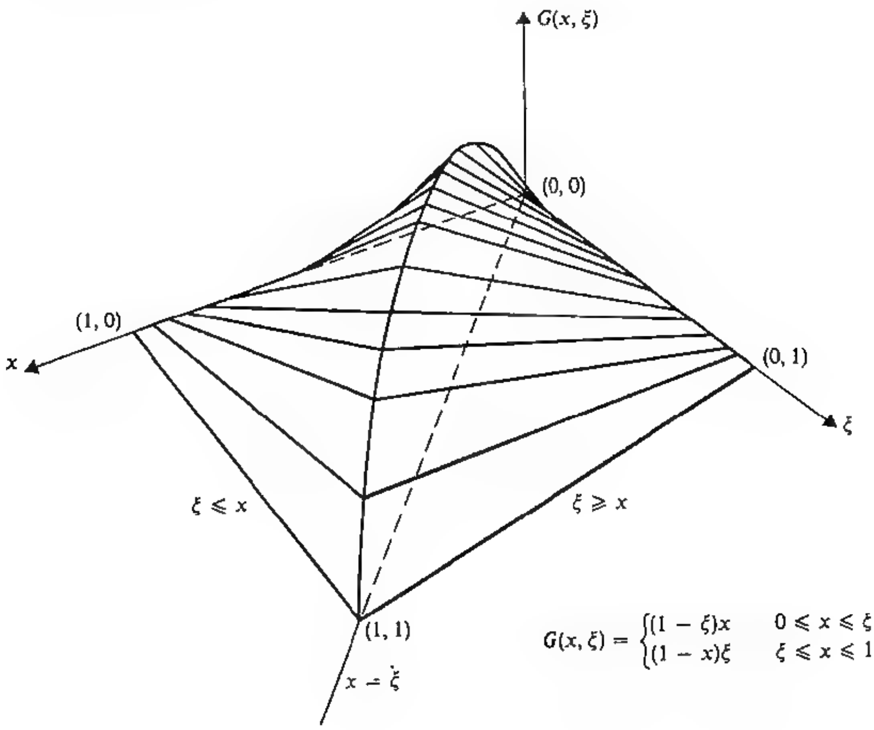
which is the same as $G(x, \xi)$.

(iv) *W: page 122, Equations (28.4)*
Note that here "d/dx" really means "$\partial/\partial x$, ξ being kept constant"; ξ has the status of a fixed parameter in these equations, which are satisfied by the function of one variable

$$x \longmapsto G(x, \xi) \qquad x \in [\alpha, \beta].$$

These equations will from now on be adopted as an alternative *definition* of G. They serve also as a practical way of obtaining G without remembering (if you ever could) the formula (28.2). The conditions (a) to (d) *should* be remembered; they follow directly from the definition of G, as you will be asked to show in SAQ 6. It can be shown that these equations are sufficient to determine G, if it exists. In particular, this implies that G is independent of any particular choice of v_1 and v_2.

(v) *W: page 122*, **Example**
We give a picture of the Green's function for this case. It is clearly its own mirror-image in the plane $x = \xi$.



$$G(x, \xi) = \begin{cases} (1 - \xi)x & 0 \leqslant x \leqslant \xi \\ (1 - x)\xi & \xi \leqslant x \leqslant 1 \end{cases}$$

SAQ 5

Complete the "algebraic manipulation" required to establish *W: page 121, lines −7, −6.*

(Solution on p. 26.)

SAQ 6

Confirm that for each $\xi \in (\alpha, \beta)$, G, defined by Equation (28.2), satisfies the conditions (28.4) in *W: page 122.*

(Solution on p. 26.)

SAQ 7

Consider the system

$$u'' + u = 0 \qquad \text{in } (0, \tfrac{1}{2}\pi),$$
$$u(0) = u(\tfrac{1}{2}\pi) = 0.$$

Show that there is no solution except the trivial solution $u = 0$. Find the Green's function for the system and confirm its symmetry.

(Solution on p. 27.)

SAQ 8

Consider the system

$$u'' + u = 0 \qquad \text{in } (0, \pi),$$
$$u(0) = u(\pi) = 0.$$

Show that there is a solution, and that no Green's function is defined by the boundary value problem (28.4) in *W: page 122*. Check also that $D = 0$ for the system.

(Solution on p. 28.)

SAQ 9

(i) Let

$$D(f, g) = f(\alpha)g(\beta) - f(\beta)g(\alpha)$$

for any two functions f and g on $[\alpha, \beta]$. Let $\{v_1, v_2\}$ and $\{v_3, v_4\}$ be any two pairs of linearly independent solutions of the equation

$$(pv')' + qv = 0 \qquad \text{in } (\alpha, \beta).$$

Prove that

$$D(v_1, v_2) = 0 \Leftrightarrow D(v_3, v_4) = 0,$$

so that the conditions $D = 0$ and $D \neq 0$ (*W: page 121, line 11*) are independent of the choice of solutions used.

(ii) Prove that if the system

$$(pu')' + qu = 0,$$
$$u(\alpha) = u(\beta) = 0,$$

has a nontrivial solution, then $D = 0$.

(iii) Prove that if the system

$$(pu')' + qu = 0,$$
$$u(\alpha) = u(\beta) = 0,$$

has no solution (except for $u \equiv 0$), then $D \neq 0$.

(iv) Deduce the converses of (ii) and (iii).

(Solution on p. 28.)

It follows from SAQ 9 that the construction of the Green's function G is possible for a boundary value problem if and only if the associated homogeneous problem has no nontrivial solution.

9.2.2 Nonhomogeneous Boundary Conditions

In the next reading passage we see how to tackle nonhomogeneous boundary conditions. We then have a brief discussion which enables us to define a Green's function where the boundary conditions involve derivatives. Finally, an example is treated.

READ W: page 123, line 3 to page 126 (the end of the section).

(i) *W: page 123, lines 5 and 10*
The notations

$$\frac{\partial G}{\partial \xi}(\alpha, \alpha) \quad \text{and} \quad \frac{\partial G}{\partial \xi}(\beta, \beta)$$

are ambiguous, and if care is not taken wrong results may follow. We illustrate the correct process as follows.

We have

$$\frac{\partial G}{\partial \xi}(x, \xi) = \begin{cases} \dfrac{\partial G^+}{\partial \xi}(x, \xi) = \dfrac{1}{KD}[v_1'(\xi)v_2(\alpha) - v_2'(\xi)v_1(\alpha)] & \\ \qquad \times [v_1(x)v_2(\beta) - v_2(x)v_1(\beta)] & x > \xi, \\ \dfrac{\partial G^-}{\partial \xi}(x, \xi) = \dfrac{1}{KD}[v_1(x)v_2(\alpha) - v_2(x)v_1(\alpha)] & \\ \qquad \times [v_1'(\xi)v_2(\beta) - v_2'(\xi)v_1(\beta)] & x < \xi. \end{cases}$$

We now define the function of one variable which maps x to

$$\begin{aligned} \frac{\partial G}{\partial \xi}(x, \alpha) = \frac{\partial G}{\partial \xi}(x, \xi)\bigg|_{\xi=\alpha} \qquad & x \in [\alpha, \beta] \\ = \frac{\partial G^+}{\partial \xi}(x, \xi)\bigg|_{\xi=\alpha} & \\ = \frac{1}{KD}[v_1'(\alpha)v_2(\alpha) - v_2'(\alpha)v_1(\alpha)][v_1(x)v_2(\beta) - v_2(x)v_1(\beta)] & \\ = \frac{1}{p(\alpha)D}[v_1(x)v_2(\beta) - v_2(x)v_1(\beta)] & \end{aligned}$$

from the definition of K. Therefore

$$\frac{\partial G}{\partial \xi}(\alpha, \alpha) = \frac{D}{p(\alpha)D} = \frac{1}{p(\alpha)},$$

and

$$\frac{\partial G}{\partial \xi}(\beta, \alpha) = 0.$$

For

$$\frac{\partial G}{\partial \xi}(\beta, \beta) \quad \text{and} \quad \frac{\partial G}{\partial \xi}(\alpha, \beta)$$

you must be careful to use G^- rather than G^+. Concisely,

$$\frac{\partial G}{\partial \xi}(\alpha,\alpha) = \frac{\partial G^+}{\partial \xi}(x,\alpha)\bigg|_{x=\alpha} \qquad \frac{\partial G}{\partial \xi}(\beta,\alpha) = \frac{\partial G^+}{\partial \xi}(x,\alpha)\bigg|_{x=\beta}$$

$$\frac{\partial G}{\partial \xi}(\alpha,\beta) = \frac{\partial G^-}{\partial \xi}(x,\beta)\bigg|_{x=\alpha} \qquad \frac{\partial G}{\partial \xi}(\beta,\beta) = \frac{\partial G^-}{\partial \xi}(x,\beta)\bigg|_{x=\beta}.$$

(ii) *W: page 123, lines 11 to 16*
We want to use the known solution of the nonhomogeneous equation with homogeneous boundary conditions (28.1) in *W: page 120* to solve the completely nonhomogeneous system (28.5). The method used depends on the fact that

$$u_1 : x \longmapsto \frac{\partial G(x,\xi)}{\partial \xi}\bigg|_{\xi=\alpha} \qquad x \in [\alpha,\beta]$$

satisfies

$$(pu_1')' - qu_1 = 0,$$

$$u_1(\alpha) = \frac{1}{p(\alpha)} \neq 0, \quad u_1(\beta) = 0,$$

and that

$$u_2 : x \longmapsto \frac{\partial G(x,\xi)}{\partial \xi}\bigg|_{\xi=\beta} \qquad x \in [\alpha,\beta]$$

satisfies

$$(pu_2')' - qu_2 = 0,$$

$$u_2(\alpha) = 0, \quad u_2(\beta) = -\frac{1}{p(\beta)} \neq 0.$$

Since u_1 and u_2 are linearly independent, all solutions of the differential equation are of the form

$$u = u_h + Au_1 + Bu_2$$

where

$$u_h(x) = \int_\alpha^\beta G(x,\xi)f(\xi)d\xi$$

is a particular solution. In fact, the particular solution u_h satisfies the homogeneous boundary conditions, which makes it easy to adjust the constants A and B so that $u(\alpha) = a$, $u(\beta) = b$.

(iii) *W: page 124, Equations (28.6)*
The special cases obtained by making some of the constants μ_1, μ_2, σ_1 or σ_2 zero should be noted. If $\mu_1 = \mu_2 = 0$ we have *Dirichlet conditions*; if $\sigma_1 = \sigma_2 = 0$, *Neumann conditions*.

SAQ 10

Show that the system

$$u'' + 4u = 1 \qquad \text{in } (0,\pi),$$

$$u(0) = 1, \quad u(\pi) = 0$$

has no solution.

(Solution on p. 29.)

SAQ 11

Solve the system

$$u'' - u = -f \qquad \text{in } (0, 1),$$

$$u(0) = 0, \quad u'(1) = 1,$$

by obtaining first the appropriate Green's function.

(Solution on p. 29.)

SAQ 12

Confirm that, in the notation of *W: page 123, lines 4 to 10*, the Green's function of the example in *W: page 122* satisfies

$$\frac{\partial G}{\partial \xi}(0, 0) = 1, \qquad \frac{\partial G}{\partial \xi}(1, 0) = 0,$$

$$\frac{\partial G}{\partial \xi}(0, 1) = 0, \qquad \frac{\partial G}{\partial \xi}(1, 1) = -1.$$

Use this Green's function to write down the solution of the system

$$u'' = -1 \qquad \text{in } (0, 1),$$

$$u(0) = u(1) = 1,$$

and check your answer.

(Solution on p. 30.)

You have seen in the examples and SAQs that, when constructing a Green's function in practice, it is more convenient to use Equations (28.4) rather than the explicit formula (28.2). It is of interest to see how the symmetry of the Green's function follows directly from the properties stated in Equations (28.4). (In practice it is often more efficient to use symmetry instead of continuity -(28.4c)—as in the examples in *W: pages 122 and 125*.) We shall also show directly that the Green's function determined by Equations (28.4) yields the solution of the nonhomogeneous problem (28.5). The advantage of this treatment is seen when we consider the more general system (28.6), for which Weinberger does not give the explicit form of the Green's function. Again, in the next unit, when we come to consider Green's functions for partial differential equations there is a close analogy with Equations (28.4).

We first prove a simple lemma which you have seen previously in *Unit M201 25, Boundary Value Problems*.

LAGRANGE'S IDENTITY

Let L be the operator defined by

$$L : u \longmapsto (pu')' + qu$$

where u is twice continuously differentiable on some interval. Then

$$\int (uLv - vLu) = p(uv' - vu').$$

Proof

We have

$$\int (uLv - vLu) = \int [u(pv')' - v(pu')'] \qquad \text{by definition of } L$$

$$= [upv' - vpu'] - \int [u'pv' - v'pu'] \qquad \text{integrating by parts}$$

$$= p(uv' - vu').$$

You may recognize that we have essentially used this result in showing that $p(x) \times$ the Wronskian is constant over the interval (*W: page 118*) and in obtaining the criterion in *W: page 124, line 6*. We now use Lagrange's Identity to obtain two further results. The methods of proof are somewhat involved, and we are not suggesting that you should be able to reproduce them "blindfold". However, we hope that you will appreciate the ways in which we use the defining properties (28.4) for the Green's function in the midst of the manipulations.

We shall first prove symmetry for the function G defined by Equations (28.4): we shall show that

$$G(x_1, x_2) = G(x_2, x_1),$$

whenever x_1 and $x_2 \in (\alpha, \beta)$.

Proof

Let x_1 and x_2 be fixed points in $[\alpha, \beta]$ with $x_1 \leqslant x_2$ and set

$$g_1 : x \longmapsto G(x, x_1) \qquad x \in [\alpha, \beta],$$

$$g_2 : x \longmapsto G(x, x_2) \qquad x \in [\alpha, \beta].$$

We apply Lagrange's Identity to g_1 and g_2 over the interval $[\alpha, \beta]$:

$$\int_{\alpha}^{x_1} (g_1 L g_2 - g_2 L g_1) + \int_{x_1}^{x_2} (g_1 L g_2 - g_2 L g_1) + \int_{x_2}^{\beta} (g_1 L g_2 - g_2 L g_1)$$

$$= \Big[p(g_1 g_2' - g_2 g_1') \Big]_{\alpha}^{x_1 - 0} + \Big[p(g_1 g_2' - g_2 g_1') \Big]_{x_1 + 0}^{x_2 - 0} + \Big[p(g_1 g_2' - g_2 g_1') \Big]_{x_2 + 0}^{\beta}$$

where we have split the interval to account for the jump discontinuities in the integrand. By Equation (28.4a) in *W: page 122*, $Lg_1 = Lg_2 = 0$ over each interval and the left-hand side of this expression vanishes. In addition we note that

$$g_1(\alpha) = g_2(\alpha) = g_1(\beta) = g_2(\beta) = 0 \qquad \text{from (28.4b)},$$

$$\left.\begin{array}{l} g_1 \text{ and } g_2 \text{ are continuous at } x_1 \text{ and } x_2 \\ g_1' \text{ is continuous at } x_2 \\ g_2' \text{ is continuous at } x_1 \end{array}\right\} \text{from (28.4a, c).}$$

Thus, collecting terms,

$$0 = p(x_1) g_2(x_1) [g_1'(x_1 + 0) - g_1'(x_1 - 0)]$$
$$+ p(x_2) g_1(x_2) [g_2'(x_2 - 0) - g_2'(x_2 + 0)].$$

Now, by (28.4d)

$$g_i'(x_i + 0) - g_i'(x_i - 0) = -\frac{1}{p(x_i)}$$

for $i = 1, 2$. Hence,

$$g_2(x_1) = g_1(x_2)$$

or

$$G(x_1, x_2) = G(x_2, x_1),$$

as required.

We shall now take the nonhomogeneous boundary value problem,

$$(pu')' + qu = -f \quad \text{in } (\alpha, \beta)$$
$$u(\alpha) = a,$$
$$u(\beta) = b,$$

and show that its solution is given by *W: page 123, line −16* when G is defined by Equations (28.4) in *W: page 122*, assuming that both (28.4) and the above problem have a solution.

We set, for some fixed $\bar{x} \in [\alpha, \beta]$,

$$g : x \longmapsto G(x, \bar{x}) \qquad x \in [\alpha, \beta],$$

and apply Lagrange's Identity to u and g over the interval (α, β):

$$\int_\alpha^{\bar{x}} (uLg - gLu) + \int_{\bar{x}}^{\beta} (uLg - gLu) = \Big[p(ug' - gu') \Big]_\alpha^{\bar{x}-0} + \Big[p(ug' - gu') \Big]_{\bar{x}+0}^{\beta}$$

The left-hand side reduces to

$$\int_\alpha^\beta gf,$$

since $Lg = 0$ except at $\bar{x}$ and $Lu = -f$ over the interval. To evaluate the right-hand side we use the conditions

$g(\alpha) = g(\beta) = 0$ from (28.4b),

g is continuous at $\bar{x}$ from (28.4c),

u and u' are continuous at $\bar{x}$

$u(\alpha) = a, \quad u(\beta) = b.$

Thus,

$$\int_\alpha^\beta gf = -p(\alpha)ag'(\alpha) + p(\bar{x})u(\bar{x})[g'(\bar{x} - 0) - g'(\bar{x} + 0)] + p(\beta)bg'(\beta).$$

By (28.4d)

$$g'(\bar{x} - 0) - g'(\bar{x} + 0) = \frac{1}{p(\bar{x})},$$

so that

$$u(\bar{x}) = \int_\alpha^\beta gf + ap(\alpha)g'(\alpha) - bp(\beta)g'(\beta).$$

Now, G is symmetric, as we have already shown. Therefore

$$g(\xi) = G(\xi, \bar{x}) = G(\bar{x}, \xi),$$

and so

$$g'(\xi) = \frac{\partial G}{\partial \xi}(\bar{x}, \xi).$$

Hence

$$u(\bar{x}) = \int_\alpha^\beta G(\bar{x}, \xi) f(\xi) d\xi + ap(\alpha) \frac{\partial G}{\partial \xi}(\bar{x}, \alpha) - bp(\beta) \frac{\partial G}{\partial \xi}(\bar{x}, \beta),$$

for all $\bar{x} \in [\alpha, \beta]$.

SAQ 13

By proceeding as on *W: page 121*, show that a unique solution to the general non-homogeneous system (28.6) does not exist unless $D \neq 0$, where D is defined two lines below Equations (28.6).

HINT: $$\frac{d}{dx}\left(\int_{x_0}^{x} H(x, \xi) d\xi \right) = H(x, x) + \int_{x_0}^{x} \frac{\partial H}{\partial x}(x, \xi) d\xi.$$

(Solution on p. 31.)

SAQ 14 (Optional)

By proceeding as in the preceding text, show that the Green's function G defined by the system (28.7) is symmetric.

(Solution on p. 32.)

SAQ 15 (Optional)

By proceeding as in the preceding text, prove that (subject to $D \neq 0$ and the existence of a solution), the solution to the system

$$(pu')' + qu = -f \qquad \text{in } (\alpha, \beta),$$
$$u'(\alpha) = a, \quad u'(\beta) = b,$$

is given by

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi - p(\alpha) a G(x, \alpha) + p(\beta) b G(x, \beta),$$

where G is the Green's function satisfying (28.7) with $\mu_1 = -1$, $\sigma_1 = 0$, $\mu_2 = 1$, $\sigma_2 = 0$.

(Solution on p. 32.)

9.2.3 An Intuitive Interpretation of the Green's Function

The solution of the system

$$(pu')' - qu = -f \qquad \text{in } (\alpha, \beta), \tag{1}$$
$$u(\alpha) = u(\beta) = 0,$$

can be represented (when it exists) in the form

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi. \tag{2}$$

By analysing this integral we shall demonstrate that such a representation is to be expected as a consequence of the linearity of the differential operator

$$L : u \longmapsto (pu')' + qu$$

and of the linear and homogeneous nature of the boundary conditions. We shall also give a direct interpretation of the Green's function G. The argument is not rigorous, and there is no need to learn it.

Partition the interval $[\alpha, \beta]$ into a large number of small intervals $(\xi_{i-1}, \xi_i]$, $i = 1, 2, \ldots, n$, where

$$\alpha = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = \beta.$$

It will do no harm to assume that the intervals are all equal in length:

$$\xi_i - \xi_{i-1} = \Delta,$$

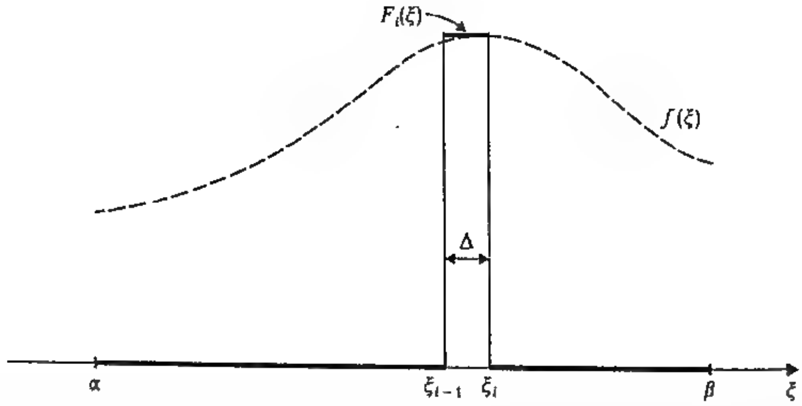
say. Then, approximately, we have from (2)

$$u(x) \simeq \sum_{i=1}^{n} G(x, \xi_i) f(\xi_i) \Delta. \tag{3}$$

We consider now the system (1) with the function f replaced by F_i, defined as

$$F_i(\xi) = \begin{cases} f(\xi_i) & \xi_{i-1} < \xi \leqslant \xi_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{4}$$

A graph of F_i is shown in the next figure.



Let the solution to Equation (1) for this case be denoted by u_i. Then Equation (2) takes the form

$$u_i(x) = \int_\alpha^\beta G(x,\xi)\, F_i(\xi)d\xi \simeq G(x,\xi_i) f(\xi_i)\Delta. \tag{5}$$

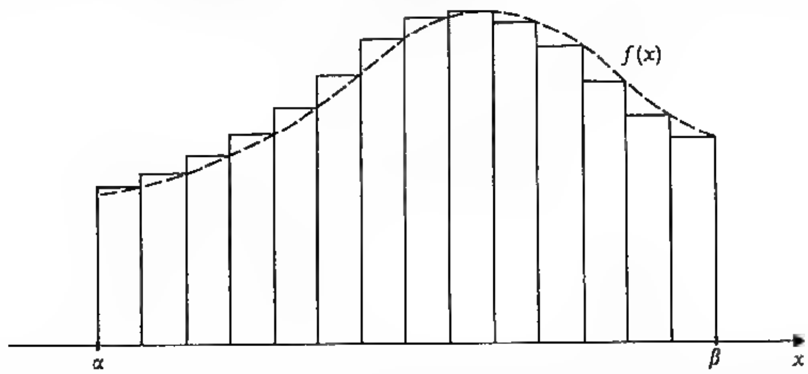
Comparing (5) with (3), we see that

$$u(x) \simeq \sum_{i=1}^{n} u_i(x). \tag{6}$$

Since

$$f(x) \simeq \sum_{i=1}^{n} F_i(x), \tag{7}$$

Equation (6) states that the solution of (1) is the sum of the solutions of several sub-problems with f replaced by the function F_i. f is built up from the F_i $(i = 1, 2, \ldots, n)$ by (7), as shown, the width of the columns being Δ.



This result is predictable from the linearity of L and the homogeneity of the boundary conditions in (1). In fact, since

$$Lu_i = (pu_i')' - qu_i = -F_i,$$
$$u_i(\alpha) = u_i(\beta) = 0,$$

for $i = 1, \ldots, n$, it follows that

$$\sum_{i=1}^{n} Lu_i = \sum_{i=1}^{n} [(pu_i')' - qu_i]$$

$$= -\sum_{i=1}^{n} F_i \simeq -f \qquad \text{from Equation (7)}$$

with

$$\sum_{i=1}^{n} u_i(\alpha) = \sum_{i=1}^{n} u_i(\beta) = 0.$$

But the first equation can be written as

$$L\sum_{i=1}^{n} u_i = \left(p\left[\sum_{i=1}^{n} u_i\right]'\right)' - q\sum_{i=1}^{n} u_i \simeq -f,$$

so that the solution of (1) satisfies (6). The validity of the argument is subject to the supposition that, by taking Δ small enough, the error in the solution of the system when using the step function approximation to f, instead of f itself, can be made arbitrarily small.

We now find the meaning of $G(x, \xi)$. We see, by applying Equation (5) to the function f given by

$$f(\xi) = \begin{cases} 1/\Delta & \xi_{i-1} < \xi \leqslant \xi_i, \\ 0 & \text{elsewhere.} \end{cases} \tag{8}$$

that $G(x, \xi_i)$ is the solution to the problem of an input function which is large in $(\xi_{i-1}, \xi_i]$ and zero elsewhere, and which has the property that

$$\int_{\alpha}^{\beta} f(\xi)d\xi = 1. \tag{9}$$

In fact f resembles the *delta function* of *Unit M201 12, Linear Functionals*, or the *impulse function* of *Unit M201 19, Laplace Transforms*. Such a concentrated input often has a physical interpretation, as in the following example.

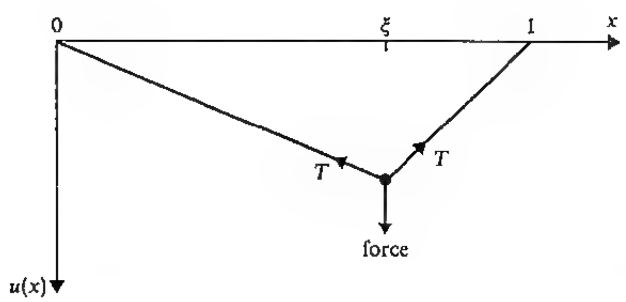
The system

$$u''(x) = -\frac{1}{T}g(x) \qquad 0 < x < 1, \tag{10}$$

$$u(0) = u(1) = 0,$$

describes the steady displacement $u(x)$ of a string stretched between $x = 0$ and $x = 1$, under a force per unit length equal to $g(x)$, where T is the tension in the string. (Equation (10) can be obtained from the vibrating string problem in *W: page 36*, by assuming there is no time variation.)

The function g may represent any kind of force distribution. We shall suppose that it represents a concentrated force such as might be produced by hanging a weight from the string at the point $x = \xi$, as shown in the figure.



There is no way of producing a force concentrated wholly at a single point—there is always a spread of force about the point $x = \xi$; but physical intuition leads us to expect that the exact distribution does not matter much, on the scale on which we are viewing the phenomenon, and that the important quantity is the magnitude of the resultant force, given by

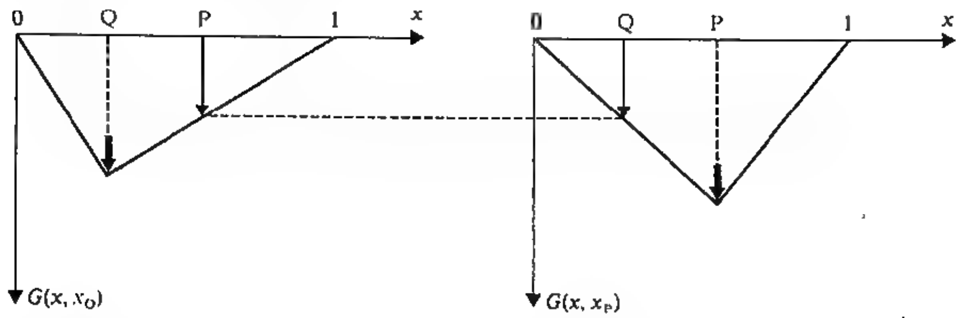
$$\int_0^1 g(x)dx.$$

Then the argument above suggests that $u(x)$ is equal to $G(x, \xi)$ where G is the Green's function for (10) (see Equations (28.4) and the example in *W: page 122*). It is obvious from the figure that the function u shares some of the properties of G: except at $x = \xi$, we have $u''(x) = 0$, u is continuous at ξ and has a discontinuity in the slope there.

The interpretation of the Green's function as the effect of a concentrated input function to the system permits a physical interpretation to be made of the symmetry of the Green's function. In the case of the string, the symmetry property,

$$G(x_P, x_Q) = G(x_Q, x_P),$$

states that the displacement at P due to a point force at Q is equal to the displacement at Q due to an equal point force at P (see figure). This *reciprocity law* is of wide importance in mathematical physics.



SAQ 16

The differential equation

$$m\ddot{x} + Kx = f$$

describes the motion of a particle of mass m attached to a spring of stiffness K and acted on by a force $f(t)$ at time t. Find the solution for $t > 0$ when $f(t)$ reduces to a unit impulse delivered at $t = 0$, the system being quiescent for $t < 0$. (A unit impulse increases the momentum of the particle (= mass × velocity) by one unit. The notations $\dot{x}$ and $\ddot{x}$ are conventionally used to denote the first and second derivatives of x with respect to t.)

(Solution on p. 33.)

9.3 SUMMARY

We have seen how ordinary differential equations may be expressed in *self-adjoint form* as

$$L[u] = f,$$

where L is the operator defined by

$$L[u] = (pu')' + qu,$$

for suitable functions p and q. We have exhibited solutions for this equation with initial and boundary conditions.

The system with *homogeneous initial values*

$$L[u] = f \qquad \text{in } (\alpha, \infty),$$
$$u(\alpha) = u'(\alpha) = 0,$$

has the solution

$$u(x) = \int_\alpha^x R(x, \xi) f(\xi) d\xi,$$

where R is the *influence function* or *one-sided Green's function*. R is given by

$$R(x, \xi) = \frac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{p(x)[v_1'(x)v_2(x) - v_2'(x)v_1(x)]},$$

where v_1 and v_2 are any linearly independent solutions of the associated homogeneous equation

$$L[v] = 0.$$

We saw that R is independent of the choice of v_1 and v_2, and that for any particular pair v_1, v_2 the denominator is constant throughout the interval.

The system with *nonhomogeneous initial values*

$$L[u] = f \qquad \text{in } (\alpha, \infty),$$
$$u(\alpha) = u_0,$$
$$u'(\alpha) = u_1,$$

has the solution

$$u(x) = \int_\alpha^x R(x, \xi) f(\xi) d\xi + c_1 v_1(x) + c_2 v_2(x),$$

where the constants c_1 and c_2 can be determined by substituting the initial conditions.

For boundary value problems we must consider the *associated homogeneous equation with homogeneous boundary conditions*:

$$\begin{aligned} L[u] &= 0 \qquad \text{in } (\alpha, \beta), \\ -\mu_1 u'(\alpha) + \sigma_1 u(\alpha) &= 0, \\ \mu_2 u'(\beta) + \sigma_2 u(\beta) &= 0. \end{aligned} \qquad (1)$$

This may have a nontrivial solution v, say. In this case cv is a solution for every constant c, and there are no other solutions. There is no Green's function (SAQ 9) and the solution of the nonhomogeneous problem does not necessarily exist; if it does exist it is not unique.

If there is no solution to the associated homogeneous problem then there is a *Green's function* G, and for each $\xi \in (\alpha, \beta)$ the function $x \longmapsto G(x, \xi)$ satisfies the following equations.

(a) $L[G] = 0 \qquad x \neq \xi,$

(b) $-\mu_1 \left.\dfrac{dG}{dx}\right|_{x=\alpha} + \sigma_1 G\Big|_{x=\alpha} = \mu_2 \left.\dfrac{dG}{dx}\right|_{x=\beta} + \sigma_2 G\Big|_{x=\beta} = 0.$

(c) $\Big[G \Big]_{x=\xi-0}^{x=\xi+0} = 0,$

(d) $\left[\dfrac{dG}{dx}\right]_{x=\xi-0}^{x=\xi+0} = -\dfrac{1}{p(\xi)}.$

G is symmetric:

$$G(x, \xi) = G(\xi, x).$$

In the case that $\mu_1 = \mu_2 = 0$, G is given explicitly by Equation (28.2) in W.

The general boundary value problem

$$\begin{aligned} L[u] &= -f \qquad \text{in } (\alpha, \beta), \\ -\mu_1 u'(\alpha) + \sigma_1 u(\alpha) &= a, \\ \mu_2 u'(\beta) + \sigma_2 u(\beta) &= b, \end{aligned}$$

in which neither $\mu_1 = \sigma_1 = 0$ nor $\mu_2 = \sigma_2 = 0$, has the solution:

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi + \frac{p(\alpha)}{\mu_1} a G(x, \alpha) + \frac{p(\beta)}{\mu_2} b G(x, \beta)$$

when $\mu_1 \neq 0$, $\mu_2 \neq 0$;

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi + \frac{p(\alpha)}{\sigma_1} a \frac{\partial G}{\partial \xi}(x, \alpha) - \frac{p(\beta)}{\sigma_2} b \frac{\partial G}{\partial \xi}(x, \beta)$$

when $\mu_1 = \mu_2 = 0$;

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi + \frac{p(\alpha)}{\mu_1} a G(x, \alpha) - \frac{p(\beta)}{\sigma_2} b \frac{\partial G}{\partial \xi}(x, \beta)$$

when $\mu_1 \neq 0$, $\mu_2 = 0$;

$$u(x) = \int_\alpha^\beta G(x, \xi) f(\xi) d\xi + \frac{p(\alpha)}{\sigma_1} a \frac{\partial G}{\partial \xi}(x, \alpha) + \frac{p(\beta)}{\mu_2} b G(x, \beta)$$

when $\mu_1 = 0$, $\mu_2 \neq 0$.

9.4 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

Solution to SAQ 1

This can be done on sight*: the self-adjoint form is

$$\frac{d}{dx}\left\{(1+x^2)\frac{du}{dx}(x)\right\} + \lambda u(x) = 0 \qquad -1 < x < 1.$$

Solution to SAQ 2

The choice

$$p(x) = \exp\left(\int_0^x \frac{d\xi}{\xi}\right)$$

will not work, because the integral does not converge (the trouble can be spotted from the fact that formally we obtain $\exp(\ln x - \ln 0)$ and $\ln 0$ is undefined). However, any indefinite integral of $1/x$ will do as the exponent; so we choose the simplest,

$$p(x) = \exp\left(\int \frac{dx}{x}\right) = \exp(\ln x) = x.$$

We now write, in the notation of *W: page 117*,

$$q(x) = \frac{p(x)c(x)}{a(x)} = -\frac{n^2}{x}.$$

$$f(x) = \frac{p(x)F(x)}{a(x)} = xF(x).$$

The self-adjoint form is then

$$\frac{d}{dx}\left(x\frac{du}{dx}(x)\right) - \frac{n^2}{x}u(x) = xF(x).$$

Solution to SAQ 3

The equation may be written

$$\frac{d}{dx}\left(1\frac{du}{dx}(x)\right) - u(x) = e^x,$$

so it is already in self-adjoint form with $p = 1$, $q = -1$, $f(x) = e^x$.

Two linearly independent solutions of the corresponding homogeneous equation are $v_1(x) = e^x$, $v_2(x) = e^{-x}$. We have

$$K = p(x)[v_1'(x)v_2(x) - v_2'(x)v_1(x)] = 2.$$

(The fact that K is constant and nonzero is a partial confirmation that the manipulations so far are right.)

Then, from Equation (27.4) in *W: page 118*

$$R(x,\xi) = \tfrac{1}{2}(e^x e^{-\xi} - e^{\xi}e^{-x}).$$

The solution is therefore

$$u(x) = \frac{1}{2}\int_0^x (e^x e^{-\xi} - e^{\xi}e^{-x})e^{\xi}\,d\xi + c_1e^x + c_2e^{-x}$$

$$= \tfrac{1}{2}(xe^x + \tfrac{1}{2}e^{-x} - \tfrac{1}{2}e^x) + c_1e^x + c_2e^{-x}.$$

The conditions $u(0) = 1$, $u'(0) = 0$ give $c_1 = c_2 = \frac{1}{2}$. The solution is therefore given by

$$u(x) = \tfrac{1}{2}xe^x + \tfrac{1}{4}e^x + \tfrac{3}{4}e^{-x}.$$

* Assuming, of course, that you have good sight.

Solution to SAQ 4

We want to find p, q, f so that the equation

$$x^2u''(x) - 2xu'(x) + 2u(x) = x^4 \qquad x > 1$$

is equivalent to the equation

$$(pu')' + qu = pu'' + p'u' + qu = f.$$

This is achieved if

$$\frac{p'(x)}{p(x)} = -\frac{2}{x}, \quad \frac{q(x)}{p(x)} = \frac{2}{x^2}, \quad \frac{f(x)}{p(x)} = \frac{x^4}{x^2} = x^2.$$

The first of these equations is satisfied by

$$p(x) = \exp\left[\int -\frac{2}{x}\,dx\right] = \frac{1}{x^2},$$

where

$$\int -\frac{2}{x}\,dx$$

could be any indefinite integral of $-2/x$, and we have naturally chosen the simplest one. Then

$$q(x) = 2/x^4 \quad \text{and} \quad f(x) = 1.$$

The self-adjoint form is therefore

$$\frac{d}{dx}\left(\frac{1}{x^2}\frac{du}{dx}(x)\right) + \frac{2}{x^4}u = 1,$$

and it has the same solution set as the original equation on $x > 1$.

We require two linearly independent solutions of the associated *homogeneous* equation, and we look for a solution of the form $x \longmapsto x^n$. Substituting $v(x) = x^n$ into

$$x^2v''(x) - 2xv'(x) + 2v(x) = 0 \qquad x > 1,$$

we obtain

$$n(n-1)x^n - 2nx^n + 2x^n = 0 \qquad x > 1.$$

If there is a value of n satisfying this equation we must have

$$\begin{aligned} 0 &= n(n-1) - 2n + 2 \\ &= (n-2)(n-1), \end{aligned}$$

that is to say $n = 2$ or $n = 1$, and so, the homogeneous differential equation has the linearly independent pair of solutions

$$v_1(x) = x, \quad v_2(x) = x^2.$$

Then, we obtain

$$p(x)[v_1'(x)v_2(x) - v_2'(x)v_1(x)] = x^{-2}[x^2 - 2x^2] = -1,$$

which is constant. Equation (27.4) in *W: page 118* gives the influence function

$$R(x, \xi) = -(x\xi^2 - \xi x^2).$$

(We can confirm that for a fixed ξ

$$R|_{x=\xi} = 0,$$

and

$$\left.\frac{dR}{dx}\right|_{x=\xi} = (-\xi^2 + 2\xi x)\Big|_{x=\xi} = \xi^2 = \frac{1}{p(\xi)}.$$

Alternatively, we could have used the homogeneous equation with these conditions to determine R.)

The solution to the given system has the form

$$\begin{aligned} u(x) &= \int_1^x (-x\xi^2 + \xi x^2)1\,d\xi + c_1x + c_2x^2 \\ &= \Big[-x(\tfrac{1}{3}\xi^3) + (\tfrac{1}{2}\xi^2)x^2\Big]_{\xi=1}^{\xi=x} + c_1x + c_2x^2 \\ &= (-\tfrac{1}{3}x^4 + \tfrac{1}{2}x^4) - (-\tfrac{1}{3}x + \tfrac{1}{2}x^2) + c_1x + c_2x^2. \end{aligned}$$

After applying the initial conditions $u(1) = 1$, $u'(1) = 0$, we obtain

$$1 = c_1 + c_2,$$

$$0 = c_1 + 2c_2;$$

therefore $c_1 = 2$, $c_2 = -1$, and finally the solution is

$$u(x) = \tfrac{7}{3}x - \tfrac{3}{2}x^2 + \tfrac{1}{6}x^4.$$

Solution to SAQ 5

The expression which we have to manipulate is

$$R(x,\xi) + \frac{v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)}{D}R(\beta,\xi).$$

Since

$$R(x,\xi) = \frac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{K}$$

and

$$D = v_1(\alpha)v_2(\beta) - v_2(\alpha)v_1(\beta) = KR(\alpha,\beta),$$

where K is a constant (*W: page 118*), this becomes

$$\begin{aligned} &\frac{1}{DK}\{[v_1(\alpha)v_2(\beta) - v_2(\alpha)v_1(\beta)][v_1(x)v_2(\xi) - v_2(x)v_1(\xi)] \\ &\qquad + [v_2(\alpha)v_1(x) - v_1(\alpha)v_2(x)][v_1(\beta)v_2(\xi) - v_2(\beta)v_1(\xi)]\} \\ &= \frac{1}{DK}[v_1(\alpha)v_1(x)v_2(\beta)v_2(\xi) + v_1(\beta)v_1(\xi)v_2(\alpha)v_2(x) \\ &\qquad - v_1(x)v_1(\xi)v_2(\alpha)v_2(\beta) - v_1(\alpha)v_1(\beta)v_2(x)v_2(\xi)]. \end{aligned}$$

It can be confirmed that this is equal to

$$-\frac{1}{KD}[v_1(\xi)v_2(\alpha) - v_2(\xi)v_1(\alpha)][v_1(x)v_2(\beta) - v_2(x)v_1(\beta)],$$

as required.

Solution to SAQ 6

We write (28.2) in the form

$$G(x,\xi) = \begin{cases} G^+(x,\xi) & \xi \leqslant x \leqslant \beta, \\ G^-(x,\xi) & \alpha \leqslant x \leqslant \xi. \end{cases}$$

(a) K and D are constants, assumed not to be zero. Equations (28.2) may be arranged to read

$$G(x,\xi) = A^+(\xi)v_1(x) + B^+(\xi)v_2(x) \qquad x \in [\xi,\beta],$$

$$G(x,\xi) = A^-(\xi)v_1(x) + B^-(\xi)v_2(x) \qquad x \in [\alpha,\xi],$$

where A^+, B^+, A^-, B^- are independent of x. Since v_1 and v_2 satisfy (28.4a), so do G^+ and G^-.

(b) Direct inspection of (28.2) shows that

$$G^+(\beta,\xi) = 0, \quad G^-(\alpha,\xi) = 0.$$

(c) Inspection of (28.2) shows that $G^-(\xi, \xi) = G^+(\xi, \xi)$. This condition states that G is continuous.

(d) We have

$$\frac{dG^+}{dx} = \frac{1}{KD}[v_1(\xi)v_2(\alpha) - v_2(\xi)v_1(\alpha)][v_1'(x)v_2(\beta) - v_2'(x)v_1(\beta)].$$

Therefore

$$\left.\frac{dG}{dx}\right|_{x=\xi+0} = \left.\frac{dG^+}{dx}\right|_{x=\xi}$$

$$= \frac{1}{KD}[v_1(\xi)v_2(\alpha) - v_2(\xi)v_1(\alpha)][v_1'(\xi)v_2(\beta) - v_2'(\xi)v_1(\beta)].$$

Similarly,

$$\left.\frac{dG}{dx}\right|_{x=\xi-0} = \left.\frac{dG^-}{dx}\right|_{x=\xi}$$

$$= \frac{1}{KD}[v_1'(\xi)v_2(\alpha) - v_2'(\xi)v_1(\alpha)][v_1(\xi)v_2(\beta) - v_2(\xi)v_1(\beta)].$$

By expanding these expressions and subtracting we find that

$$\left.\frac{dG}{dx}\right|_{x=\xi+0} - \left.\frac{dG}{dx}\right|_{x=\xi-0}$$

$$= \frac{1}{KD}[v_1'(\xi)v_2(\xi)v_1(\beta)v_2(\alpha) + v_1(\xi)v_2'(\xi)v_1(\alpha)v_2(\beta) - v_1'(\xi)v_2(\xi)v_1(\alpha)v_2(\beta)$$

$$- v_1(\xi)v_2'(\xi)v_1(\beta)v_2(\alpha)]$$

$$= \frac{1}{KD}[-v_1'(\xi)v_2(\xi) + v_1(\xi)v_2'(\xi)][v_1(\alpha)v_2(\beta) - v_1(\beta)v_2(\alpha)]$$

But the first bracket equals $-K/p(\xi)$ and the second equals D, so the result follows.

Solution to SAQ 7

The general solution is

$$u = A \sin x + B \cos x.$$

The condition $u(0) = 0$ gives $B = 0$, and $u(\frac{1}{2}\pi) = A \sin \frac{1}{2}\pi = A$ is zero only if $A = 0$ too.

To construct the Green's function it is better not to copy from (28.2), but to build it up from (28.4) in the following way.

$$G^-(x, \xi) = C(\xi) \sin x$$

satisfies $G^-(0, \xi) = 0$ and the differential equation.

$$G^+(x, \xi) = D(\xi) \cos x$$

satisfies $G^+(\frac{1}{2}\pi, \xi) = 0$ and the differential equation.

Continuity at $x = \xi$ requires

$$D(\xi) \cos \xi = C(\xi) \sin \xi.$$

The discontinuity in the derivative (since $p = 1$) is given by

$$-D(\xi) \sin \xi - C(\xi) \cos \xi = -1.$$

Therefore $C(\xi) = \cos \xi$, $D(\xi) = \sin \xi$ and

$$G(x, \xi) = \begin{cases} \sin \xi \cos x & 0 \leqslant \xi \leqslant x \leqslant \frac{1}{2}\pi, \\ \cos \xi \sin x & 0 \leqslant x \leqslant \xi \leqslant \frac{1}{2}\pi. \end{cases}$$

To prove symmetry, we interchange x and ξ (see note (iii) of Section 9.2.1), and obtain

$$G(\xi, x) = \begin{cases} \sin x \cos \xi & 0 \leqslant x \leqslant \xi \leqslant \frac{1}{2}\pi, \\ \cos x \sin \xi & 0 \leqslant \xi \leqslant x \leqslant \frac{1}{2}\pi. \end{cases}$$

which is equal to $G(x, \xi)$.

Solution to SAQ 8

A continuous solution is $u = \sin x$ (or any multiple of this).

The Green's function must satisfy the following conditions:

(a) $\dfrac{d^2G}{dx^2} + G = 0 \qquad x \neq \xi,$

(b) $G(0, \xi) = G(\pi, \xi) = 0,$

(c) $\Big[G \Big]_{\xi-0}^{\xi+0} = 0,$

(d) $\left[\dfrac{dG}{dx} \right]_{\xi-0}^{\xi+0} = -1.$

(The notation in (c) and (d) should be self-explanatory.)

Every solution of (a) on $(0, \xi)$ which is zero at $x = 0$ is of the form $A(\xi) \sin x$, and every solution on (ξ, π) which is zero at $x = \pi$ is of the form $B(\xi) \sin x$. Condition (c) requires that $A = B$, since $\sin \xi \neq 0 \quad \forall \xi \in (0, \pi)$, and therefore (d) cannot be satisfied.

Putting $v_1(x) = \sin x$ and $v_2(x) = \cos x$, we obtain

$$D = v_1(0)v_2(\pi) - v_2(0)v_1(\pi) = 0.$$

Solution to SAQ 9

(i) Any three solutions of the homogeneous equation must be linearly dependent, so

$$v_3 = av_1 + bv_2,$$

$$v_4 = cv_1 + dv_2$$

for some a, b, c, d. Therefore

$$D(v_3, v_4) = (ad - bc)D(v_1, v_2).$$

If $ad - bc = 0$, the equations above imply that v_3, v_4 are linearly dependent, contrary to hypothesis. Therefore $ad - bc \neq 0$ and the result follows.

(ii) Let u_1 be a solution of the given homogeneous system. Then $D(u_1, u_2) = 0$ for any choice of u_2, since $u_1(\alpha) = u_1(\beta) = 0$ by definition. It follows from (i) that $D = 0$ whatever solutions are used.

(iii) Let u_1, u_2 be two linearly independent solutions of the homogeneous equation, satisfying respectively

$$u_1(\alpha) = 0, \quad u_2(\beta) = 0.$$

Then $u_1(\beta) \neq 0$, $u_2(\alpha) \neq 0$, or the given condition would be violated. Now

$$D(u_1, u_2) = u_1(\alpha)u_2(\beta) - u_1(\beta)u_2(\alpha)$$
$$= -u_1(\beta)u_2(\alpha) \neq 0.$$

Therefore, by (i), $D \neq 0$ for every choice of solutions.

(iv) The converses state that if, for some choice of solutions of the homogeneous *equation*, $D = 0$, then there exists a solution of the homogeneous *system*—for if there were no such solution, $D \neq 0$ by (iii); and if $D \neq 0$, there is no solution of the homogeneous system—for if there were a solution, $D = 0$ by (ii).

Solution to SAQ 10

The associated homogeneous problem

$$v'' + 4v = 0 \qquad \text{in } (0, \pi),$$
$$v(0) = v(\pi) = 0,$$

has the nontrivial solution $v: x \longmapsto \sin 2x$. Hence the given system will have solutions only if the criterion in *W: page 124, line 6* is satisfied. For our problem the criterion requires

$$2 = \int_0^{\pi} \sin 2x \, dx,$$

which is untrue.

Solution to SAQ 11

The Green's function G must satisfy the Equations (28.7) which, for our problem, reduce to

$$\frac{d^2G}{dx^2} - G = 0 \qquad x \neq \xi, \tag{1}$$

$$G\Big|_{x=0} = \frac{dG}{dx}\Big|_{x=1} = 0, \tag{2}$$

$$G\big|_{x=\xi+0} - G\big|_{x=\xi-0} = 0, \tag{3}$$

$$\frac{dG}{dx}\Big|_{x=\xi+0} - \frac{dG}{dx}\Big|_{x=\xi-0} = -1. \tag{4}$$

A linearly independent pair of solutions of (1) is $\{e^x, e^{-x}\}$. The differential equation and conditions (2) are satisfied by

$$G^+(x, \xi) = A(\xi)(e^{-1}e^x + ee^{-x}) \qquad x \geqslant \xi,$$
$$G^-(x, \xi) = B(\xi)(e^x - e^{-x}) \qquad x \leqslant \xi.$$

Equation (3) requires that

$$A(\xi)(e^{\xi-1} + e^{1-\xi}) - B(\xi)(e^{\xi} - e^{-\xi}) = 0, \tag{5}$$

and Equation (4) gives

$$A(\xi)(e^{\xi-1} - e^{1-\xi}) - B(\xi)(e^{\xi} + e^{-\xi}) = -1. \tag{6}$$

Equations (5) and (6) give

$$A(\xi) = \frac{e^{\xi} - e^{-\xi}}{2(e + e^{-1})}, \quad B(\xi) = \frac{e^{\xi-1} + e^{1-\xi}}{2(e + e^{-1})},$$

and the Green's function is given by

$$G(x, \xi) = \begin{cases} G^+(x, \xi) = \dfrac{(e^{\xi} - e^{-\xi})(e^{x-1} + e^{1-x})}{2(e + e^{-1})} & x \geqslant \xi, \\[2ex] G^-(x, \xi) = \dfrac{(e^{\xi-1} + e^{1-\xi})(e^x - e^{-x})}{2(e + e^{-1})} & x \leqslant \xi. \end{cases} \tag{7}$$

This can now be substituted in the appropriate version of the last paragraph of *W: page 124*. However, it is most unlikely that you will remember such a formula, and the following procedure is better in this case.

Let

$$g(x) = \int_0^1 G(x, \xi) f(\xi) d\xi,$$

so that

$$g'' - g = -f \qquad \text{in } (0, 1).$$

Since

$$G\Big|_{x=0} = \frac{dG}{dx}\Big|_{x=1} = 0$$

for every $\xi \in (0, 1)$ (by definition), g must satisfy

$$g(0) = g'(1) = 0.$$

Therefore we have to add to g some function h which satisfies the conditions

$$h'' - h = 0 \qquad \text{in } (0, 1),$$
$$h(0) = 0, \quad h'(1) = 1.$$

The function $x \longmapsto G^-(x, 1)$ satisfies

$$\frac{\partial^2 G^-}{\partial x^2}(x, 1) - G^-(x, 1) = 0,$$
$$G^-(0, 1) = 0,$$

and

$$\frac{\partial G^-}{\partial x}(1, 1) = 1.$$

Therefore,

$$h(x) = G^-(x, 1) = \frac{e^x - e^{-x}}{e + e^{-1}}$$

and

$$u(x) = \int_0^1 G(x, \xi) f(\xi) d\xi + \frac{e^x - e^{-x}}{e + e^{-1}}$$
$$= \int_0^x G^+(x, \xi) f(\xi) d\xi + \int_x^1 G^-(x, \xi) f(\xi) d\xi + \frac{e^x - e^{-x}}{e + e^{-1}},$$

where G^+ and G^- are given by Equation (7).

Solution to SAQ 12

We have $p(x) = 1$, and

$$G(x, \xi) = \begin{cases} G^+(x, \xi) = \xi(1 - x) & x \geqslant \xi, \\ G^-(x, \xi) = (1 - \xi)x & x \leqslant \xi. \end{cases}$$

Now,

$$\frac{\partial G}{\partial \xi}(0, 0) = \frac{\partial G^+}{\partial \xi}(x, 0)\Big|_{x=0};$$

we therefore consider

$$\frac{\partial G^+}{\partial \xi}(x, 0) = 1 - x,$$

and letting $x = 0$ this becomes 1, which is equal to $1/p(0)$.

Similarly,

$$\frac{\partial G}{\partial \xi}(1, 0) = 1 - 1 = 0.$$

Next, we require

$$\frac{\partial G}{\partial \xi}(1, 1) = \frac{\partial G^-}{\partial \xi}(x, 1)\Big|_{x=1}$$

Therefore, we consider

$$\frac{\partial G^-}{\partial \xi}(x, 1) = -x,$$

which gives us

$$\frac{\partial G}{\partial \xi}(1, 1) = -1 = -\frac{1}{p(1)},$$

and

$$\frac{\partial G}{\partial \xi}(0, 1) = 0.$$

Using the formula (*W: page 123, line 16*), we obtain

$$\begin{aligned} u(x) &= \int_0^x (1 - x)\xi \, d\xi + \int_x^1 x(1 - \xi)d\xi + 1 \,.\, 1 \,.\, (1 - x) - 1 \,.\, 1 \,.\, (-x) \\ &= (1 - x)\tfrac{1}{2}x^2 + x[(1 - x) - \tfrac{1}{2}(1 - x^2)] + 1 \\ &= -\tfrac{1}{2}x^2 + \tfrac{1}{2}x + 1. \end{aligned}$$

Check: $u''(x) = -1$, $u(0) = 1$, $u(1) = -\frac{1}{2} + \frac{1}{2} + 1 = 1$.

Solution to SAQ 13

The function

$$-\int_\alpha^x R(x, \xi) f(\xi) d\xi$$

is one solution of the equation $(pu')' + qu = -f$, though it does not necessarily satisfy the given boundary conditions. Every solution is therefore of the form

$$u(x) = -\int_\alpha^x \frac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{K} f(\xi)d\xi + c_1v_1(x) + c_2v_2(x),$$

where we have written $R(x, \xi)$ out in full as on *W: page 118*, and we wish to choose c_1, c_2 so that the boundary conditions of (28.6) are satisfied.

Using the differentiation formula in the hint, we have

$$\begin{aligned} u'(x) &= -\frac{v_1(x)v_2(x) - v_2(x)v_1(x)}{K} f(x) - \int_\alpha^x \frac{v_1'(x)v_2(\xi) - v_2'(x)v_1(\xi)}{K} f(\xi)d\xi \\ &\qquad + c_1v_1'(x) + c_2v_2'(x) \\ &= -\int_\alpha^x \frac{v_1'(x)v_2(\xi) - v_2'(x)v_1(\xi)}{K} f(\xi)d\xi + c_1v_1'(x) + c_2v_2'(x). \end{aligned}$$

Therefore,

$$\begin{aligned} u(\alpha) &= c_1v_1(\alpha) + c_2v_2(\alpha), \\ u'(\alpha) &= c_1v_1'(\alpha) + c_2v_2'(\alpha), \\ u(\beta) &= -\int_\alpha^\beta \frac{v_1(\beta)v_2(\xi) - v_2(\beta)v_1(\xi)}{K} f(\xi)d\xi + c_1v_1(\beta) + c_2v_2(\beta), \\ u'(\beta) &= -\int_\alpha^\beta \frac{v_1'(\beta)v_2(\xi) - v_2'(\beta)v_1(\xi)}{K} f(\xi)d\xi + c_1v_1'(\beta) + c_2v_2'(\beta). \end{aligned}$$

The boundary conditions therefore become, after some rearrangement:

$$\begin{aligned} &c_1[-\mu_1v_1'(\alpha) + \sigma_1v_1(\alpha)] + c_2[-\mu_1v_2'(\alpha) + \sigma_1v_2(\alpha)] = a, \\ &c_1[\mu_2v_1'(\beta) + \sigma_2v_1(\beta)] + c_2[\mu_2v_2'(\beta) + \sigma_2v_2(\beta)] \\ &\qquad = b + \int_\alpha^\beta \frac{[\mu_2v_1'(\beta) + \sigma_2v_1(\beta)]v_2(\xi) - [\mu_2v_2'(\beta) + \sigma_2v_2(\beta)]\, v_1(\xi)}{K} f(\xi)d\xi. \end{aligned}$$

These equations have a unique solution c_1, c_2 provided the determinant D of the coefficients is not zero. If

$$D = 0,$$

a solution may exist, but is not then unique.

Solution to SAQ 14

Let x_1 and x_2 be fixed points in $[\alpha, \beta]$ with $x_1 \leqslant x_2$ and set

$$g_1 : x \longmapsto G(x, x_1) \qquad x \in [\alpha, \beta],$$

$$g_2 : x \longmapsto G(x, x_2) \qquad x \in [\alpha, \beta].$$

We proceed as in Section 9.2.2, noting that the boundary conditions are now, using (28.7b),

$$-\mu_1 g_i'(\alpha) + \sigma_1 g_i(\alpha) = \mu_2 g_i'(\beta) + \sigma_2 g_i(\beta) = 0$$

for $i = 1, 2$. We obtain

$$\begin{aligned} 0 = {} & -p(\alpha)[g_1(\alpha)g_2'(\alpha) \quad g_2(\alpha)g_1'(\alpha)] + p(x_1)g_2(x_1)[g_1'(x+0) - g_1'(x-0)] \\ & + p(x_2)g_1(x_2)[g_2'(x-0) - g_2'(x+0)] + p(\beta)[g_1(\beta)g_2'(\beta) - g_2(\beta)g_1'(\beta)] \end{aligned}$$

and when we "plug in" the boundary conditions the first and last brackets vanish. Finally, the jump condition (28.7d) gives

$$g_i'(x_i + 0) - g_i'(x_i - 0) = -\frac{1}{p(x_i)}$$

for $i = 1, 2$; so that

$$g_2(x_1) = g_1(x_2)$$

or

$$G(x_1, x_2) = G(x_2, x_1),$$

and we have proved symmetry.

Solution to SAQ 15

For some fixed $\bar{x} \in [\alpha, \beta]$ we set

$$g : x \longmapsto G(x, \bar{x}) \qquad x \in [\alpha, \beta],$$

and apply Lagrange's Identity as in Section 9.2.2, noting that the boundary conditions are now

$$g'(\alpha) = g'(\beta) = 0$$

$$u'(\alpha) = a, \quad u'(\beta) = b.$$

Thus we obtain

$$\int_\alpha^\beta gf = p(\alpha)g(\alpha)a + p(\bar{x})u(\bar{x})[g'(\bar{x} - 0) - g'(\bar{x} + 0)] - p(\beta)g(\beta)b.$$

By (28.7d)

$$g'(\bar{x} - 0) - g'(\bar{x} + 0) = \frac{1}{p(\bar{x})},$$

so that

$$u(\bar{x}) = \int_\alpha^\beta gf - p(\alpha)ag(\alpha) + p(\beta)bg(\beta).$$

Now G is symmetric, as shown in the solution to SAQ 14. Therefore

$$g(\xi) = G(\xi, \bar{x}) = G(\bar{x}, \xi).$$

Hence

$$u(\bar{x}) = \int_{\alpha}^{\beta} G(\bar{x}, \xi) f(\xi) d\xi - p(\alpha) a G(\bar{x}, \alpha) + p(\beta) b G(\bar{x}, \beta),$$

for all $\bar{x} \in [\alpha, \beta]$.

Solution to SAQ 16

The velocity $\dot{x}$ immediately after the unit impulse is delivered is given by

$$m\dot{x}(0 + 0) = 1,$$

and the displacement is still zero, $x(0 + 0) = 0$. Subsequently the displacement is governed by the system

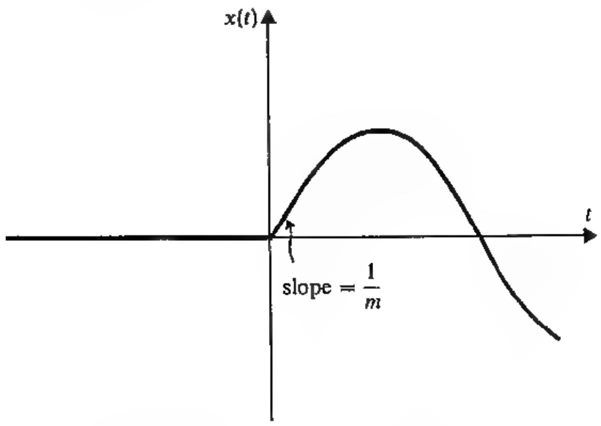
$$\left.\begin{aligned} m\ddot{x} + Kx = 0 \qquad \text{in } R^{+}, \\ x(0) = 0, \quad \dot{x}(0) = \frac{1}{m}. \end{aligned}\right\} \quad (1)$$

The general solution of the equation is

$$A \sin \left(\frac{K}{m}\right)^{\frac{1}{2}} t + B \cos \left(\frac{K}{m}\right)^{\frac{1}{2}} t,$$

and the initial conditions give

$$B = 0 \quad \text{and} \quad A \left(\frac{K}{m}\right)^{\frac{1}{2}} = \frac{1}{m}.$$



The system defining R on *W: page 119* is analogous to the system (1). This question elucidates the remark preceding the first example on *W: page 119*.

Unit 10 Green's Functions II: Partial Differential Equations

Contents

Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).
H. F. Weinberger, *A First Course in Partial Differential Equations* (Xerox, 1965).

It is essential to have these books: the course is based on them and will not make sense without them. They are referred to in the text as ***S*** and ***W*** respectively.

Unit 10 is based on *W*: Chapter IV, Section 24 and Chapter V, Sections 29 and 30.

Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics*. References to Open University courses in mathematics take the form.

> *Unit M100 13, Integration II* for the Mathematics Foundation Course.
> *Unit M201 23, The Wave Equation* for the Linear Mathematics Course.

10.0 INTRODUCTION

We begin this unit by looking at Laplace's equation in a circle, solutions for which we shall require in the sections following. We go on to extend the ideas of *Unit 9, Green's Functions I*, where Green's functions were introduced for ordinary differential equations, and solve nonhomogeneous *partial* differential equations in two dimensions. Our aim is achieved, in the first instance, by introducing the *finite Fourier transform*.

We then try a different approach using *δ-functions* in order that we may relate our results to the definition of the one-dimensional Green's function in *Unit 9* and use analogous definitions in two and three dimensions. The Green's function treatment is used finally to treat problems with nonhomogeneous boundary conditions, using Poisson's equation in two dimensions to illustrate the techniques.

10.1 THE FINITE FOURIER TRANSFORM

10.1.1 Laplace's Equation in a Circle

READ W: Section 24, pages 100 to 103, ignoring page 102, line 5 and the reference to Exercise 3 at the end.

Notes

(i) *W: page 100, line 8*
In order to conform with the usage in *W*, we shall use log for the natural logarithm function from now on.

(ii) *W: page 100, line −5*
A singular point of the ordinary differential equation

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

is a point x where either $a(x) = 0$, or $b(x)$ or $c(x)$ is infinite. Thus, a normal differential equation has no singular points.

(iii) *W: page 101, lines 8 to 10*
Uniform convergence and term-by-term differentiation were discussed in *Unit 6, Fourier Series*

SAQ 1

W: page 103, Exercise 1

(Solution on p. 23.)

SAQ 2

W: page 104, Exercise 10

(Solution on p. 23.)

10.1.2 Nonhomogeneous Problems

READ W: Section 29, page 126 to page 127, line −5, then page 128, line −7 to page 130, line 11 and page 130, line −11 to page 131, the end of the section.

The omitted passages include two complicated verifications and a reference to an earlier result in *W* which we have not asked you to read: for the purposes of this course the effort required to read these passages would not be worthwhile

Notes

(i) *W: page 127, line −9*
This means solving the differential equation by the integrating factor method.

(ii) *W: page 129, Equation (29.8)*
This equation could represent the case of *steady state* heat conduction, with heat absorbed within a disc of radius R at a steady state rate $F(r, \theta)$ per unit time per unit area.

(iii) *W: page 129, lines 9 and10*
This was discussed in Section 10.1.1 (*W: page 100*).

(iv) *W: page 130, Equations (29.9)*
You will be asked to derive these equations in SAQ 3. It turns out however that for the problem under discussion the associated homogeneous equation has solutions which are unbounded as $r \rightsquigarrow 0$. As a result the theory of Green's functions developed in *Unit 9* must be modified by replacing the boundary condition

$$G|_{r=0} = 0$$

by

$$G|_{r \rightsquigarrow 0} \text{ is bounded.}$$

We shall not prove this result.

SAQ 3

Solve the problem

$$(ra_n')' - \frac{n^2}{r}a_n = -rA_n \qquad 0 < r < R,$$

$$a_n(R) = 0,$$

$$a_n(r) \text{ is bounded as } r \rightsquigarrow 0,$$

where A_n is a specified function.

HINT: Treat the cases $n = 0$ and $n \neq 0$ separately. The general solution to the associated homogeneous equation may be found in *W: page 100*.

(Solution on p. 24.)

General Comment

In the discussion of the nonhomogeneous heat conduction problem, the first term on the right-hand side (*W: page 127, line 6*) vanishes when we put in $u(0, t) = u(\pi, t) = 0$. Now suppose that the boundary conditions too are nonhomogeneous, say

$$u(0, t) = f(t),\ u(\pi, t) = g(t).$$

The term in square brackets would not vanish, resulting in

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}(x, t)\sin nx\,dx = \frac{2}{\pi}[-g(t)n\cos n\pi + f(t)n] - n^2 b_n(t)$$

and (29.3) would be replaced by

$$b_n'(t) + n^2 b_n(t) - B_n(t) + \frac{2}{\pi}[(-1)^{n+1}ng(t) + nf(t)].$$

This is *still* a linear first-order differential equation in one variable of the same type as (29.3), so that nonhomogeneous boundary conditions are still amenable to the finite sine transform method of solution. The solution corresponding to *W: page 127, line −5* for nonhomogeneous boundary conditions is

$$u(x, t) \sim \sum_1^\infty \int_0^t e^{-n^2(t-\tau)}\left\{B_n(\tau) + \frac{2}{\pi}[(-1)^{n+1}ng(\tau) + nf(\tau)]\right\} d\tau \sin nx.$$

Example

Solve the system

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = x && 0 < x < \pi,\ t > 0,\\
&u(0, t) = 0 && t \geqslant 0,\\
&\frac{\partial u}{\partial x}(\pi, t) = 1 && t \geqslant 0, \qquad (1)\\
&u(x, 0) = 0 && 0 < x < \pi.
\end{aligned}$$

Solution

We consider first the associated homogeneous problem

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \qquad 0 < x < \pi, t > 0,$$

$$v(0, t) = 0 \qquad t \geqslant 0,$$

$$\frac{\partial v}{\partial x}(\pi, t) = 0 \qquad t \geqslant 0,$$

$$v(x, 0) = 0 \qquad 0 < x < \pi.$$

Using separation of variables, we write $v(x, t) = X(x)T(t)$. Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \text{ say,}$$

where λ is a constant. The solution of

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X'(\pi) = 0,$$

is

$$X(x) = C_1 \sin \sqrt{\lambda}x + C_2 \cos \sqrt{\lambda}x,$$

with $C_2 = 0$ and $\lambda = (n + \frac{1}{2})^2 \quad (n = 0, 1, 2, \ldots)$ to satisfy the boundary conditions. Thus we obtain the eigenfunctions

$$X_n(x) = \sin(n + \tfrac{1}{2})x \quad \text{for } n \geqslant 0.$$

We can expand any piecewise smooth function g in the form*

$$g(x) = \sum_{n=0}^{\infty} d_n \sin(n + \tfrac{1}{2})x$$

on $0 < x < \pi$, with d_n given by

$$d_n = \frac{\displaystyle\int_0^\pi g(x)\sin(n + \tfrac{1}{2})x\,dx}{\displaystyle\int_0^\pi \sin^2(n + \tfrac{1}{2})x\,dx}$$

$$= \frac{2}{\pi}\int_0^\pi g(x)\sin(n + \tfrac{1}{2})x\,dx.$$

We can now see more clearly what type of Fourier series will enable us to tackle our problem (1). The associated homogeneous system suggests an expansion in terms of the basis $\{\sin(n + \frac{1}{2})x\}$ *whatever the right-hand sides of (1) may be, provided they are suitably well behaved.* Note, incidentally, that this is the *only* purpose we have in looking at the homogeneous system—it has no other relevance whatsoever.

Now we can proceed as in *W: Section 29* and write

$$u(x, t) \sim \sum_{n=0}^{\infty} b_n(t)\sin(n + \tfrac{1}{2})x, \tag{2}$$

where

$$b_n(t) = \frac{2}{\pi}\int_0^\pi u(x, t)\sin(n + \tfrac{1}{2})x\,dx,$$

and take the finite sine transform

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}(x, t)\sin(n + \tfrac{1}{2})x\,dx = \frac{2}{\pi}\left[\frac{\partial u}{\partial x}(x, t)\sin(n + \tfrac{1}{2})x - u(x, t)(n + \tfrac{1}{2})\cos(n + \tfrac{1}{2})x\right]_0^\pi$$
$$- (n + \tfrac{1}{2})^2 \frac{2}{\pi}\int_0^\pi u(x, t)\sin(n + \tfrac{1}{2})x\,dx.$$

*A method of proving this statement is suggested in Exercise 3 of *W: Section 20.*

The crucial requirement is that the term in brackets on the right-hand side can be evaluated. The particular choice of $\sin(n + \frac{1}{2})x$, derived from the associated homogeneous system, makes this possible.

We have, in this case

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}(x,t)\sin(n+\tfrac{1}{2})x\,dx = \frac{2}{\pi}\sin(n+\tfrac{1}{2})\pi - (n+\tfrac{1}{2})^2\frac{2}{\pi}\int_0^\pi u(x,t)\sin(n+\tfrac{1}{2})x\,dx$$
$$= (-1)^n\frac{2}{\pi} - (n+\tfrac{1}{2})^2 b_n(t),$$

$$\frac{2}{\pi}\int_0^\pi \frac{\partial u}{\partial t}(x,t)\sin(n+\tfrac{1}{2})x\,dx = \frac{db_n}{dt}(t),$$

and

$$\frac{2}{\pi}\int_0^\pi x\sin(n+\tfrac{1}{2})x\,dx = \frac{2}{\pi}\frac{(-1)^n}{(n+\frac{1}{2})^2}.$$

Thus

$$\frac{db_n}{dt} + (n+\tfrac{1}{2})^2 b_n = (-1)^n\frac{2}{\pi}\left[1 + \frac{1}{(n+\frac{1}{2})^2}\right],$$

which has the general solution

$$b_n(t) = A_n e^{-(n+\frac{1}{2})^2 t} + \frac{(-1)^n}{(n+\frac{1}{2})^2}\cdot\frac{2}{\pi}\left[1 + \frac{1}{(n+\frac{1}{2})^2}\right].$$

We have, from (2) together with the initial condition,

$$0 = u(x,0) = \sum_{n=0}^{\infty} b_n(0)\sin(n+\tfrac{1}{2})x,$$

so that $b_n(0) = 0$.

Therefore,

$$A_n = -\frac{(-1)^n}{(n+\frac{1}{2})^2}\cdot\frac{2}{\pi}\left[1 + \frac{1}{(n+\frac{1}{2})^2}\right]$$

and

$$u(x,t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})^2}\left[1 + \frac{1}{(n+\frac{1}{2})^2}\right][1 - e^{-(n+\frac{1}{2})^2 t}]\sin(n+\tfrac{1}{2})x.$$

We note, of course, that the *individual* terms in the series are NOT solutions of the original partial differential equation.

SAQ 4

Use the finite Fourier transform method to solve the system

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \sin x \qquad 0 < x < \pi,\ t > 0,$$
$$u(0,t) = u(\pi,t) = 0 \qquad t \geqslant 0,$$
$$u(x,0) = 1 \qquad 0 < x < \pi.$$

(Solution on p. 25.)

SAQ 5

Use the finite Fourier transform method to solve the following system (with $0 < k < 1$):

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1 \qquad 0 < x < \pi,\ t > 0,$$
$$u(0,t) = u(\pi,t) = 0 \qquad t \geqslant 0,$$
$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \qquad 0 < x < \pi.$$

This is a damped wave equation, usually called the *equation of telegraphy*.

(Solution on p. 26.)

SAQ 6

Use the finite Fourier transform method to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0 \qquad \text{in } (0, \pi) \times (0, \pi),$$

$$u(0, y) = u(\pi, y) = 0 \qquad y \in [0, \pi],$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, \pi) = 1 \qquad x \in (0, \pi).$$

(Solution on p. 28.)

SAQ 7

A rudimentary, one-dimensional model of a nuclear reactor is as follows. There is a thin rod of active material of length a, and the reaction in it is governed by the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta^2 u \qquad 0 < x < a, \ t > 0,$$

where $u(x, t)$ is the neutron density and α and β are positive constants: the neutron density is taken to be zero at $x = 0$, $x = a$. The term $\beta^2 u$ describes the rate of generation of fresh neutrons, and the term $\alpha^2 \partial^2 u/\partial x^2$ describes the diffusion of neutrons. Show that there is a critical length of rod $\pi\alpha/\beta$ below which, in general, the reaction dies away, and above which it rapidly increases.

(Solution on p. 29.)

10.2 GREEN'S FUNCTION IN TWO-DIMENSIONAL DOMAINS

READ W, Section 30, page 132 to page 133, line −8.

Notes

(i) *W: page 132, lines −5 to −3*
You should carry this out: the new variable ϕ has appeared from writing

$$A_n(\rho) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\rho, \phi)\cos n\phi \, d\phi$$

$$B_n(\rho) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\rho, \phi)\sin n\phi \, d\phi$$

(*W: page 129, lines −9 and −8* — the names of the variables *must* be changed, since r and θ are in use already in the first equation of *W: Section 30*).

(ii) *W: page 133, line 3*
The sense is made clearer if the words "uniformly in ϕ" are deleted.

(iii) *W: page 133, Equation (30.4)*
Having derived this equation we use it in (30.2), putting

$$z^n = \left(\frac{R}{r}\right)^n \left(\frac{\rho}{R}\right)^n = \left(\frac{\rho}{r}\right)^n,$$

i.e.

$$z = \frac{\rho}{r},$$

and, also

$$z^n = \left(\frac{r}{R}\right)^n \left(\frac{\rho}{R}\right)^n = \left(\frac{r\rho}{R^2}\right)^n,$$

i.e.

$$z = \frac{r\rho}{R^2}.$$

(iv) *W: page 133, Equation (30.5)*
To obtain this from the previous equation, we write

$$\frac{1}{2\pi}\log\frac{R}{r} = \frac{1}{4\pi}\log(R^2) - \frac{1}{4\pi}\log(r^2)$$

and then combine the terms.

The next passage in *W* is a verification that the expression

$$u(r, \theta) = \int_0^R \int_{-\pi}^{\pi} G(r, \theta; \rho, \phi)F(\rho, \phi)\rho \, d\rho \, d\phi$$

does indeed satisfy the system (29.8). It might not have done so, since a priori there is no guarantee either that the system has a solution, or that such manipulations as the expansion of $u(r, \theta)$ as a Fourier series and the interchange of summation and integration leading to (30.2) are, in fact, legitimate. There is always the possibility that the whole operation is a purely formal exercise. The proof of validity has been omitted from the reading passage, since the analysis is somewhat involved, but it is important to realize that mechanical procedures do not always give the right answers. Here are two examples to underline what we mean.

Example 1

Find the real roots of

$$x^2 + x + 1 = 0 \tag{1}$$

(there are none!).

"*Solution*"

Re-write the equation as

$$x(x + 1) + 1 = 0.$$

But from (1)

$$x + 1 = -x^2,$$

so (1) becomes

$$-x^3 + 1 = 0, \tag{2}$$

which has $x = 1$ as a root. This, however, is not a solution of the original equation. [Note that $x^3 - 1 = (x^2 + x + 1)(x - 1)$, so solutions of (1) are also solutions of (2), but by increasing the degree of the polynomial we have inevitably added another (spurious) root.]

Example 2

Find all the solutions of the equation

$$\frac{dy}{dx} = 2y^{\frac{1}{2}} \qquad x \in R.$$

"*Solution*"

Separation of variables gives, provided $y \neq 0$ for all $x \in R$,

$$\int \frac{dy}{2y^{\frac{1}{2}}} = \int dx + c,$$

i.e.

$$y^{\frac{1}{2}} = x + c, \text{ or } y = (x + c)^2,$$

where c is arbitrary. This represents a family of parabolas, but the left-hand half of any such parabola has a negative slope and satisfies

$$\frac{dy}{dx} = -2y^{\frac{1}{2}},$$

and so is not a solution of the original equation. Thus, we have obtained non-existent "solutions" as well as genuine ones.

Thus we should realize that it is important to ask the question "Why should our derivation of the solution be any more reliable than these processes?" and to be aware that it can be answered satisfactorily.

SAQ 8

Show that

$$(r, \theta) \longmapsto \log\left[R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta - \phi)\right]$$

satisfies Laplace's equation for all (r, θ) and that

$$(r, \theta) \longmapsto \log[r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)]$$

satisfies Laplace's equation for $(r, \theta) \neq (\rho, \phi)$. Hence verify the statement in *W: page 133, line −11* that $(r, \theta) \longmapsto G(r, \theta; \rho, \phi)$ satisfies Laplace's equation except at the point (ρ, ϕ).

(Solution on p. 30.)

READ W: page 135, line 7 to page 136, line −3.

Notes

(i) *W: page 135, lines 10 to 12*
Since D is bounded it is contained in some circle $\bar{D}$, of radius R, say. Thus, for any point $(r, \theta) \in D$,

$$u(r, \theta) = \iint_{\bar{D}} G(r, \theta; \rho, \phi) F(\rho, \phi) \rho \, d\rho \, d\phi$$

$$= \iint_{\bar{D}} \frac{1}{4\pi} \left\{ -\log[r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)] + \log\left[R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta - \phi) \right] \right\} F(\rho, \phi) \rho \, d\rho \, d\phi$$

satisfies

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -F.$$

Now in forming $\nabla^2 u$, we can differentiate the second term in the braces under the integral sign; this term will then vanish, since it is harmonic on the whole of the region $\bar{D}$, as we have seen in SAQ 8. Similarly, the first term vanishes under ∇^2 throughout that part of $\bar{D}$ not also in D. However, this term has a singularity at $r = \rho$, $\theta = \phi$ and we cannot use the same procedure in D. Therefore, the function

$$v(r, \theta) = \frac{1}{4\pi} \iint_D \log[r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)] F(\rho, \phi) \rho \, d\rho \, d\phi$$

satisfies

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \nabla^2 u = -F$$

throughout D.

The condition in W that F be continuously differentiable is required so that the given formula for u satisfies Poisson's equation.

(ii) *W: page 136, Equation (30.11)*
This equation separates the "singular" part of the Green's function, reduced to its simplest form, from the smooth (harmonic) part, which is denoted by γ.

(iii) *W: page 136, line −11*
We require $\nabla^2 \gamma = 0$, and are given the form of γ on the boundary $r = R$. We know that

$$r^n \sin n\theta$$

and

$$r^n \cos n\theta$$

satisfy Laplace's equation in the circle $0 \leqslant r < R$, and can therefore write down the form of the solution γ by multiplying the nth term of the series by that multiple of r^n which yields the correct value on the boundary.

SAQ 9

W: page 140, Exercise 1

(Solution on p. 31.)

SAQ 10

W: page 140, Exercise 2

(Solution on p. 32.)

SAQ 11

W: page 140, Exercise 5

(Solution on p. 34.)

10.3 GREEN'S FUNCTION: AN ALTERNATIVE APPROACH

10.3.0 Introduction

In this section, we seek an approach to Green's function which relates closely to Equations (28.4) in *W: page 122*. Accordingly, we return to the one-dimensional Green's function and show how its definition can be expressed in terms of the δ-function; this definition is then extended to more than one dimension.

Finally, we apply Green's function methods to Poisson's equation in two dimensions.

10.3.1 The Delta Function

In *Unit 9* we encountered the idea of an impulse function: somewhat analogously, in *Unit 7, Overhead Wires* we made use of a force per unit length which is zero except at the point of application of a force (at which point the force per unit length is not defined). In a similar way, we may wish to represent a point mass by a density function which is zero everywhere except at the mass, where it is "infinite". To permit a mathematical description of such physical situations, we use the symbol $\delta(x - \xi)$ to denote a quantity which is zero everywhere except at ξ, where it is not defined, and such that

$$\int_{\alpha}^{\beta} f(x)\delta(x-\xi)\,dx = \begin{cases} f(\xi) & \xi \in (\alpha,\beta) \\ 0 & \xi \notin (\alpha,\beta) \end{cases} \qquad (1)$$

for any function f whose domain includes (α, β). The object $\delta(x - \xi)$ is known as a **delta-function** (or **δ-function**) and you have met it before in *Unit M201 12, Linear Functionals*: it is not really a function but a *generalized function* or *distribution*. It is important to realize that no meaning can be attached to $\delta(x - \xi)$ *except in terms of integration.*

The idea of a delta function can be extended to higher dimensions, and we can introduce a generalized function $\delta(\mathbf{r} - \boldsymbol{\rho})$ in three dimensions such that

$$\begin{aligned} &\delta(\mathbf{r}-\boldsymbol{\rho}) = 0 && \mathbf{r} \neq \boldsymbol{\rho} \\ &\iiint_D f(\mathbf{r})\delta(\mathbf{r}-\boldsymbol{\rho})\,dV = \begin{cases} f(\boldsymbol{\rho}) & \boldsymbol{\rho} \in D \\ 0 & \boldsymbol{\rho} \notin D \end{cases} \end{aligned} \qquad (2)$$

where f is a function whose domain includes D. (Try to construct the corresponding definition for the two-dimensional δ-function.) No confusion should arise out of using the symbol δ for the delta function in different dimensional domains.

It is possible to express $\delta(\mathbf{r} - \boldsymbol{\rho})$ in terms of the one-dimensional δ-function introduced in Equation (1). If $\mathbf{r}$ is expressed in Cartesian coordinates, then we replace dV in the integral by $dx\,dy\,dz$, and you can easily verify by using an arbitrary function f that

$$\delta(\mathbf{r}-\boldsymbol{\rho}) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta),^{*}$$

where $\boldsymbol{\rho}$ is the vector (ξ, η, ζ). Similarly, $\mathbf{r}$ could be expressed in spherical or cylindrical polar coordinates. For example, if $\mathbf{r}$ is expressed in spherical polar coordinates as (r, θ, ϕ) and if $\boldsymbol{\rho}$ is (ρ, ψ, ν), then we replace dV in the integral by $r^2 \sin\theta\,dr\,d\theta\,d\phi$, so that

$$\delta(\mathbf{r}-\boldsymbol{\rho}) = \frac{\delta(r-\rho)\delta(\theta-\psi)\delta(\phi-\nu)}{r^2\sin\theta}^{*} \qquad (3)$$

provided that $\rho \neq 0$. If $\rho = 0$, we have

$$\delta(\mathbf{r}) = \frac{\delta(r)}{4\pi r^2}.^{*}$$

* In expressions like this the equals sign means that, when placed in an arbitrary integral over a domain (with the correct dimension), both sides yield the same result.

We can verify this result by looking at (2) and integrating over a sphere:

$$\int\int\int_D f(\mathbf{r})\frac{\delta(r)}{4\pi r^2}r^2\sin\theta\,dr\,d\theta\,d\phi = \int\int\int_D f(r,\theta,\phi)\frac{\delta(r)}{4\pi}\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{1}{4\pi}f(0)\int_0^{\pi}\int_0^{2\pi}\sin\theta\,d\theta\,d\phi$$

$$= f(0)$$

$$= \int\int\int_D f(\mathbf{r})\delta(\mathbf{r})\,dV.$$

SAQ 12

Find an expression for $\delta(\mathbf{r}-\boldsymbol{\rho})$ similar to (3) in plane polar coordinates with $\mathbf{r} = (r, \theta)$ and $\boldsymbol{\rho} = (\rho, \psi)$. What is $\delta(\mathbf{r})$?

(Solution on p. 35.)

10.3.2 Application to Differential Equations

To find a solution to the ordinary differential equation

$$L[u](x) = -f(x) \qquad x \in (\alpha, \beta) \tag{4}$$

where L is the linear differential operator given by

$$L : u \longmapsto (pu')' + qu$$

we construct the Green's function G. A particular solution of (4) is now given by

$$u(x) = \int_\alpha^\beta f(\xi)G(x,\xi)\,d\xi,$$

so that

$$L[u](x) = L\int_\alpha^\beta f(\xi)G(x,\xi)\,d\xi.$$

Note that, since L operates on x only, it can be moved inside the integral. Comparing with (4), we have

$$\int_\alpha^\beta f(\xi)L[G](x,\xi)\,d\xi = -f(x).$$

Also we know that

$$L[G](x,\xi) = 0 \quad \text{for } x \neq \xi.$$

Thus we see that

$$L[G](x,\xi) = -\delta(x-\xi). \tag{5}$$

Indeed, we can use this equation as the basis of an alternative *definition* of the Green's function; namely for each $\xi \in (\alpha, \beta)$ we specify $x \longmapsto G(x,\xi)$ as the continuous solution of Equation (5) which also satisfies the homogeneous boundary conditions

$$G(\alpha,\xi) = G(\beta,\xi) = 0.$$

This is in fact another way of writing Equations (28.4) in *W* because G is a solution of

$$[pG']' + qG = 0 \qquad \text{for } x \neq \xi,$$

and integrating (5) over the small interval $(\xi-\varepsilon, \xi+\varepsilon)$ we find that

$$\Big[pG'\Big]_{\xi-\varepsilon}^{\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} qG = -1,$$

using the integration property of the δ-function. In the limit as $\varepsilon \rightsquigarrow 0$,

$$G'|_{\xi+0} - G'|_{\xi-0} = -\frac{1}{p(\xi)}$$

in accordance with Equation (28.4d) in W. Thus the δ-function provides a convenient notation for expressing the two conditions (28.4a) and (28.4d).

SAQ 13

Suppose H is given such that for each $\xi \in R$ the function

$$x \longmapsto H(x, \xi) \qquad x \in R$$

is a (continuous) solution of

$$(pH')' + qH = -\delta(x - \xi) \qquad x \in R$$

and, for each $x \in R$,

$$\xi \longmapsto H(x, \xi) \qquad \xi \in R$$

is continuous. Show that

$$u(x) = \int_{-\infty}^{\infty} H(x, \xi) f(\xi)\, d\xi$$

is a particular solution of

$$(pu')' + qu = -f \quad \text{in } R.$$

(Solution on p. 35.)

Example

Find G satisfying

$$G''(t, \tau) + \omega^2 G(t, \tau) = -\delta(t - \tau) \qquad t \in R, \tag{6}$$

$$G(t, \tau) = 0 \qquad t \leqslant \tau,$$

for each $\tau \in R$.

Hence, show that a solution of the ordinary differential equation

$$u'' + \omega^2 u = -f \qquad t \in R,$$

is given by

$$u(t) = -\int_{-\infty}^{t} f(\tau) \frac{\sin \omega(t - \tau)}{\omega}\, d\tau.$$

Solution

The general solution of

$$G'' + \omega^2 G = 0$$

is $B \sin(\omega t + \beta)$. We determine β from the condition $G(\tau, \tau) = 0$. Thus

$$G(t, \tau) = B \sin \omega(t - \tau).$$

Now, integration of the differential equation (6) from $\tau - \varepsilon$ to $\tau + \varepsilon$ yields, as $\varepsilon \rightsquigarrow 0$,

$$G'(\tau + 0, \tau) - G'(\tau - 0, \tau) = -1,$$

i.e.,

$$\omega B = -1 \quad \text{or} \quad B = -\frac{1}{\omega}.$$

Hence

$$G(t, \tau) = \begin{cases} \dfrac{-\sin \omega(t - \tau)}{\omega} & t \geqslant \tau, \\ 0 & t \leqslant \tau. \end{cases}$$

Using the result of SAQ 13 a particular solution of the nonhomogeneous equation is

$$u(t) = \int_{-\infty}^{\infty} G(t, \tau) f(\tau)\, d\tau$$

$$= -\int_{-\infty}^{t} f(\tau) \frac{\sin \omega(t - \tau)}{\omega}\, d\tau.$$

The generalization of these results to higher dimensions is now disarmingly simple. In analogy with (4), we should look at

$$L[u] = -F \quad \text{in } D, \tag{7}$$

where, to be specific, D is a two-dimensional domain, and show that

$$u(\mathbf{r}) = \iint_D G(\mathbf{r}, \boldsymbol{\rho})F(\boldsymbol{\rho})\,d\boldsymbol{\rho} \qquad \mathbf{r} \in D, \tag{8}$$

is a solution, where $G(\mathbf{r}, \boldsymbol{\rho})$ is a (continuous) solution of

$$L[G](\mathbf{r}, \boldsymbol{\rho}) = -\delta(\mathbf{r} - \boldsymbol{\rho}) \qquad \mathbf{r} \in D. \tag{9}$$

(We have used $d\boldsymbol{\rho}$ in place of the more usual dA to stress that $\boldsymbol{\rho}$, not $\mathbf{r}$, is the variable of integration.)

SAQ 14

Verify that (8) is a solution of (7).

(Solution on p. 35.)

10.3.3 Poisson's Equation

We have seen in Section 10.3.2 that if

$$L[G](\mathbf{r}, \boldsymbol{\rho}) = -\delta(\mathbf{r} - \boldsymbol{\rho}) \qquad \mathbf{r} \in D \tag{9}$$

for some linear differential operator L then a solution of

$$L[u] = -F \qquad \text{in } D$$

is given by

$$u(\mathbf{r}) = \iint_D G(\mathbf{r}, \boldsymbol{\rho})F(\boldsymbol{\rho})\,d\boldsymbol{\rho}. \tag{8}$$

Clearly, if G is chosen as that solution of (9) which also satisfies the boundary condition

$$G(\mathbf{r}, \boldsymbol{\rho}) = 0 \qquad \text{for } \mathbf{r} \in C$$

where C is the boundary of D, then u (as defined above) also satisfies

$$u(\mathbf{r}) = 0 \qquad \text{for } \mathbf{r} \in C.$$

In this case G as defined here is just the *Green's function* which we met in Section 10.2.

We show how to derive the form of G directly for the case when L is the Laplacian operator ∇^2. We must solve, for each $\boldsymbol{\rho} \in D$,

$$\begin{aligned} \nabla^2 G(\mathbf{r}, \boldsymbol{\rho}) &= -\delta(\mathbf{r} - \boldsymbol{\rho}) \qquad \mathbf{r} \in D, \\ G(\mathbf{r}, \boldsymbol{\rho}) &= 0 \qquad\qquad\quad \mathbf{r} \in C. \end{aligned} \tag{10}$$

For a given $\boldsymbol{\rho} \in D$ we can choose a disc $\bar{D}$, which is wholly contained in D, such that $\boldsymbol{\rho} \in \bar{D}$.

We now choose the point $\boldsymbol{\rho}$ as the origin of a new set of polar coordinates $(\bar{r}, \bar{\theta})$. Changing the origin does not change the Laplacian operator (this is seen best by looking at the definition of ∇^2 in Cartesian coordinates and noting that it is not altered by a change in the origin). Let

$$g(\bar{r}) = G(\mathbf{r}, \boldsymbol{\rho})$$

where

$$\bar{\mathbf{r}} = \mathbf{r} - \boldsymbol{\rho},$$

i.e. $\bar{\mathbf{r}}$ gives the coordinates of $\mathbf{r}$ with respect to the new origin $\boldsymbol{\rho}$. Then integrating over the disc $\bar{D}$ we have

$$\iint_{\bar{D}} \nabla^2 g(\bar{\mathbf{r}})\, dA = -\iint_{\bar{D}} \delta(\bar{\mathbf{r}})\, dA = -1,$$

since $\mathbf{0} \in \bar{D}$. To evaluate the left-hand side of this equation we use the Divergence Theorem (*W: page 53*) for the vector field **grad** g: thus

$$\iint_{\bar{D}} \nabla^2 g = \iint_{\bar{D}} \operatorname{div} \mathbf{grad}\, g$$

$$= \oint_{\bar{C}} \mathbf{n} \cdot \mathbf{grad}\, g$$

where $\bar{C}$ is the circle bounding $\bar{D}$ and $\mathbf{n}$ is a unit outward normal. We have

$$\mathbf{n} \cdot \mathbf{grad}\, g = \frac{\partial g}{\partial \bar{r}}$$

since $\bar{C}$ is a circle whose centre is at the origin.

We now require a solution of Equation (10), which must satisfy

$$\frac{\partial^2 g}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial g}{\partial \bar{r}} + \frac{1}{\bar{r}^2} \frac{\partial^2 g}{\partial \bar{\theta}^2} = 0 \qquad \text{for } \bar{r} \neq 0,$$

subject to the additional property that

$$\oint_{C} \frac{\partial g}{\partial \bar{r}} = -1. \tag{11}$$

Let us look first for solutions of the homogeneous differential equation which are independent of $\bar{\theta}$, i.e. $\partial g/\partial \bar{\theta} = 0$. These may be shown to be of the form

$$g(\bar{r}, \bar{\theta}) = A \log \bar{r} + B.$$

Then

$$\frac{\partial g}{\partial \bar{r}} = \frac{A}{r_0} \qquad \text{on } \bar{C}$$

where r_0 is the radius of $\bar{C}$ and

$$\oint_{C} \frac{\partial g}{\partial \bar{r}} = \frac{A}{r_0} \oint_{C} ds = \frac{A}{r_0} \int_0^{2\pi} r_0\, d\bar{\theta} = 2\pi A.$$

From Equation (11) we require

$$A = -\frac{1}{2\pi},$$

so that

$$-\frac{1}{2\pi} \log \bar{r}$$

is a particular solution of

$$\nabla^2 g(\bar{r}) = -\delta(\bar{\mathbf{r}}).$$

Now let γ be the solution of

$$\nabla^2 \gamma = 0 \qquad \text{in } D,$$

$$\gamma(\mathbf{r}; \boldsymbol{\rho}) = \frac{1}{2\pi} \log \bar{r} \qquad \mathbf{r} \in C,$$

where $\bar{r}$ is the distance from $\boldsymbol{\rho}$ to $\mathbf{r}$: then we recover the result in *W: page 136* that the Green's function is given by

$$G(\mathbf{r}; \boldsymbol{\rho}) = -\frac{1}{4\pi} \log(\bar{r}^2) + \gamma(\mathbf{r}; \boldsymbol{\rho}).$$

In polar coordinates,

$$\bar{r}^2 = r^2 + \rho^2 - 2r\rho\cos(\theta - \phi);$$

in rectangular Cartesian coordinates,

$$\bar{r}^2 = (x - \xi)^2 + (y - \eta)^2,$$

where (ξ, η) gives the coordinates of ρ.

If we are given a domain D which is a proper subset of R^2, then we can often obtain the Green's function for ∇^2 on D by the following argument (much as in SAQ 9). Let P be the "source" point $\rho \in D$, Q a point not in D and T the point $\mathbf{r} \in D$. Now, any multiple of log QT satisfies Laplace's equation throughout D (since $Q \notin D$) and so does a constant. We therefore seek a suitable choice of Q and suitable values for $A(\rho)$ and $B(\rho)$ such that

$$\begin{aligned} G(\mathbf{r}; \rho) &= \frac{1}{2\pi}[-\log PT + A(\rho)\log QT + B(\rho)] \\ &= \frac{1}{2\pi}\left[\log\frac{(QT)^{A(\rho)}}{PT} + B(\rho)\right] \end{aligned}$$

vanishes on C. In particular we try to choose Q so that QT/PT is independent of $\mathbf{r}$ when $T \in C$.

Example

Find Green's function for Poisson's equation in the half plane $x \geqslant 0$.

Solution

We need $G(0, y; \xi, \eta) = 0$. If P is the point (ξ, η), Q is $(-\xi, \eta)$ and T is (x, y), then PT = QT when T is on the boundary $x = 0$, and we choose

$$G(x, y; \xi, \eta) = \frac{1}{2\pi}\log\frac{QT}{PT},$$

so that $G = 0$ on $x = 0$ is satisfied. Then

$$G(x, y; \xi, \eta) = \frac{1}{4\pi}\log\left[\frac{(x + \xi)^2 + (y - \eta)^2}{(x - \xi)^2 + (y - \eta)^2}\right].$$

SAQ 15 (Optional)

Let O be the centre of a circle of radius R. Let OPQ be a straight line and let P and Q be such that OQ . OP $= R^2$. Show that the ratio SP/SQ does not vary as S describes the circumference of the circle.

(Solution on p. 36.)

SAQ 16

Use SAQ 15 to find the Green's function for Poisson's equation in the circle $0 < r < R, 0 \leqslant \theta < 2\pi$.

(Solution on p. 36.)

We can use the Green's function to solve the general nonhomogeneous Poisson's equation as follows. We seek the solution (in two dimensions) of

$$\begin{aligned} \nabla^2 u &= -F \quad \text{in } D, \qquad (12) \\ u &= f \quad \text{on } C. \end{aligned}$$

In SAQ 11(c) of *Unit 3, Elliptic and Parabolic Equations* we obtained the general form of Green's Theorem,

$$\iint_D (u\nabla^2 v - v\nabla^2 u)\, dA = \oint_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds,$$

for any (differentiable) scalar fields u and v. Let u be the solution of our problem (12) and let

$$v : (x, y) \longmapsto G(x, y; x_0, y_0)$$

for some fixed point $(x_0, y_0) \in D$.

Then Green's Theorem gives us

$$\iint_D (u\nabla^2 G - G\nabla^2 u)\, dA = \oint_C \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) ds.$$

Now, we have

$$\nabla^2 G(\rho : \mathbf{r}_0) = -\delta(\rho - \mathbf{r}_0) \qquad \text{in } D,$$

where $\mathbf{r}_0 = (x_0, y_0)$,

$$\nabla^2 u = -F \qquad \text{in } D,$$

$$u = f \qquad \text{on } C$$

and

$$G = 0 \qquad \text{on } C.$$

Hence the equation above reduces to

$$-\iint_D u(\rho)\delta(\rho - \mathbf{r}_0)\, d\rho + \iint_D G(\rho : \mathbf{r}_0)F(\rho)\, d\rho = \oint_C f(\rho)\frac{\partial G}{\partial n}(\rho : \mathbf{r}_0)\, ds,$$

i.e.

$$u(\mathbf{r}_0) = \iint_D G(\mathbf{r}_0 : \rho)F(\rho)\, d\rho - \oint_C \frac{\partial G}{\partial n}(\mathbf{r}_0 : \rho)f(\rho)\, ds,$$

using the symmetry of the Green's function (which, to be rigorous, we should have proved). In Cartesian coordinates we write this as

$$u(x, y) = \iint_D G(x, y : \xi, \eta)F(\xi, \eta)\, d\xi\, d\eta - \oint_C \frac{\partial G}{\partial n}(x, y : \xi, \eta)f(\xi, \eta)\, ds \qquad (x, y) \in D.$$

Note that $\partial G/\partial n$ denotes the derivative in the direction of the outward normal (on C) of the function

$$(\xi, \eta) \longmapsto G(x, y : \xi, \eta).$$

SAQ 17

Obtain Poisson's integral formula (*W: page 103*) for the solution of Laplace's equation in a circle,

$$\nabla^2 u = 0 \qquad \text{in the circle } r < R,$$

$$u(R, \theta) = f(\theta) \qquad 0 \leqslant \theta < 2\pi,$$

using the methods of this section.

(Solution in *W: page 139, lines −10 to −7.*)

SAQ 18

Use the Green's function obtained in the text to show that, if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x > 0,$$

$$u(0, y) = f(y) \qquad y \in R,$$

then a solution is given by

$$u(x, y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(\eta)\, d\eta}{x^2 + (y - \eta)^2}.$$

(Solution on p. 37.)

10.4 SUMMARY

In this unit we have extended Green's functions to partial differential equations in two dimensions. The finite Fourier transform was introduced as a method for reducing a nonhomogeneous partial differential equation to a set of ordinary differential equations, and shown to lead to Green's function for Poisson's equation.

We then demonstrated how Green's function could be defined using δ-functions. Poisson's equation in two dimensions was treated in this way with nonhomogeneous boundary conditions, making use of the Divergence Theorem and Green's Theorem. The δ-function was defined by

$$\delta(\mathbf{r} - \boldsymbol{\rho}) = 0 \qquad \mathbf{r} \neq \boldsymbol{\rho},$$

$$\int_D f(\mathbf{r})\delta(\mathbf{r} - \boldsymbol{\rho})\, d\mathbf{r} = \begin{cases} f(\boldsymbol{\rho}) & \boldsymbol{\rho} \in D, \\ 0 & \boldsymbol{\rho} \notin D, \end{cases}$$

for all functions f. (This definition may be interpreted in one or more dimensions.)

We saw that the Green's function for the problem

$$Lu = -F \qquad \text{in } D$$

is given by the continuous solution of

$$LG(\mathbf{r}; \boldsymbol{\rho}) = -\delta(\mathbf{r} - \boldsymbol{\rho}) \qquad \mathbf{r} \in D$$

$$G(\mathbf{r}; \boldsymbol{\rho}) = 0 \qquad \mathbf{r} \in C$$

for each $\boldsymbol{\rho} \in D$, where C is the boundary of D.

10.5 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

Solution to SAQ 1

We need to express $\sin^3\theta$ in the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n\cos n\theta + b_n\sin n\theta).$$

Now,

$$i\sin\theta = \tfrac{1}{2}(e^{i\theta} - e^{-i\theta}),$$

so, taking the cube of each side, we obtain

$$-i\sin^3\theta = \tfrac{1}{8}(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}),$$

i.e.

$$\begin{aligned}\sin^3\theta &= -\tfrac{1}{8}(2\sin 3\theta - 6\sin\theta)\\ &= \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta. \qquad (1)\end{aligned}$$

For $r \leqslant 1$, the solution is given by Equation (24.6) in *W: page 101*,

$$u(r,\theta) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta),$$

with the coefficients given by

$$u(1,\theta) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_n\cos n\theta + b_n\sin n\theta).$$

Comparing with (1), we see that

$$u(r,\theta) = \tfrac{1}{4}(3r\sin\theta - r^3\sin 3\theta) \qquad 0 \leqslant r \leqslant 1.$$

Solution to SAQ 2

To solve

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \qquad 1 < r < R,\ 0 < \theta < 2\pi,$$

$$u(1,\theta) = 0,$$

$$u(R,\theta) = f(\theta),$$

we put $u(r,\theta) = w(r)\Theta(\theta)$.

Then the equation becomes

$$\frac{r^2 w''(r)}{w(r)} + \frac{rw'(r)}{w(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = n^2, \text{ say.}$$

We now have $r^2w'' + rw' - n^2w = 0$, whose general solution is

$$w_0(r) = a_0 + b_0\log r \qquad n = 0,$$

$$w_n(r) = a_nr^n + b_nr^{-n} \qquad n \neq 0,$$

and $\Theta'' + n^2\Theta = 0$, with the eigenfunctions (for solutions with period 2π)

$$1,\ \theta \qquad n = 0,$$

$$\cos n\theta,\ \sin n\theta \qquad n \in Z^+.$$

The boundary condition $u(1,\theta) = 0$ is equivalent to $w(1) = 0$, so that

$$w_0(r) = b_0\log r,$$

$$w_n(r) = b_n(r^{-n} - r^n) \qquad n \in Z^+.$$

Hence we seek a series solution in the form

$$u(r,\theta) = b_0\log r + \sum_{n=1}^{\infty} (r^{-n} - r^n)(c_n\cos n\theta + d_n\sin n\theta).$$

To satisfy the boundary condition at $r = R$ we require that

$$f(\theta) = b_0 \log R + \sum_{n=1}^{\infty} (R^{-n} - R^n)(c_n \cos n\theta + d_n \sin n\theta) \qquad \theta \in [0, 2\pi).$$

Thus

$$b_0 \log R = \tfrac{1}{2} A_0$$
$$(R^{-n} - R^n) c_n = A_n \qquad n \in Z^+$$
$$(R^{-n} - R^n) d_n = B_n \qquad n \in Z^+,$$

where A_n, B_n are the Fourier coefficients

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \cos n\phi \, d\phi \qquad n \geqslant 0,$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \sin n\phi \, d\phi \qquad n > 0.$$

We consider the function

$$u(r, \theta) = \tfrac{1}{2} A_0 \frac{\log r}{\log R} + \sum_{n=1}^{\infty} \frac{r^{-n} - r^n}{R^{-n} - R^n} (A_n \cos n\theta + B_n \sin n\theta)$$
$$1 \leqslant r \leqslant R,\ 0 \leqslant \theta < 2\pi.$$

If

$$C = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(\theta)| \, d\theta,$$

then $|A_n| \leqslant C$, $|B_n| \leqslant C$, and the series for u and its first and second partial derivatives are dominated by

$$\sum_{n=1}^{\infty} 2Cn^2 \frac{r^n}{R^n}$$

and, hence, by the convergent series

$$\sum_{n=1}^{\infty} 2Cn^2 \frac{r_0^n}{R^n}$$

for $1 \leqslant r \leqslant r_0$ where $r_0 < R$. Thus, by Weierstrass' M-test the series for $u(r, \theta)$ and its first and second partial derivatives are uniformly convergent for $1 \leqslant r \leqslant r_0$. Hence we may differentiate the series term by term and verify that u satisfies Laplace's equation for $1 < r < R$, and also u is continuous for $1 \leqslant r < R$. By an argument analogous to that given in *W: page 101, lines −13 to −4* we may show that it is in fact continuous for $1 \leqslant r \leqslant R$ and so the sum of the series given above for $u(r, \theta)$ is the solution to our problem.

Solution to SAQ 3

To solve the problem we construct the Green's function G, which must satisfy the following equations for each $\rho \in (0, R)$:

$$(rG')' - \frac{n^2}{r} G = 0 \qquad r \neq \rho, \tag{1}$$

$$G|_{r \to 0} \text{ is bounded}, \quad G|_{r=R} = 0, \tag{2}$$

$$G(r, \rho) = G(\rho, r), \tag{3}$$

$$G'|_{r=\rho+0} - G'|_{r=\rho-0} = -\frac{1}{\rho}. \tag{4}$$

Note that we have replaced the continuity condition by the symmetry condition as the latter is easier to apply.

For the case $n = 0$ a linearly independent pair of solutions of (1) is $\{1, \log r\}$. To satisfy (2) we set

$$G_0(r, \rho) = \begin{cases} C_1(\rho)\,(\log r - \log R) & r \geqslant \rho, \\ C_2(\rho) & r \leqslant \rho. \end{cases}$$

Condition (3) now gives us

$$C_1(\rho) = C,$$
$$C_2(\rho) = C(\log \rho - \log R),$$

where C is some constant which we determine from (4), obtaining

$$C = -1.$$

Thus the Green's function is

$$G_0(r, \rho) = \begin{cases} \log R - \log r & r \geqslant \rho, \\ \log R - \log \rho & r \leqslant \rho, \end{cases}$$

and the solution to the problem is

$$a_0(r) = \int_0^R G_0(r, \rho)\rho A_0(\rho)\,d\rho$$
$$= \int_0^r \log\frac{R}{r}\,\rho A_0(\rho)\,d\rho + \int_r^R \log\frac{R}{\rho}\,\rho A_0(\rho)\,d\rho \qquad 0 \leqslant r \leqslant R.$$

For $n \neq 0$ we may assume without loss of generality that $n > 0$. A linearly independent pair of solutions is r^n, r^{-n}, and to satisfy the boundary conditions (2) we require

$$G_n(r, \rho) = \begin{cases} D_1(\rho)\left[\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n\right] & r \geqslant \rho, \\ D_2(\rho)r^n & r \leqslant \rho. \end{cases}$$

Condition (3) now gives us

$$D_1(\rho) = D\rho^n,$$
$$D_2(\rho) = D\left[\left(\frac{R}{\rho}\right)^n - \left(\frac{\rho}{R}\right)^n\right],$$

and we determine the constant D from (4), obtaining

$$D = \frac{1}{2nR^n}.$$

Thus the Green's function is

$$G_n(r, \rho) = \begin{cases} \dfrac{1}{2n}\left(\dfrac{\rho}{R}\right)^n\left[\left(\dfrac{R}{r}\right)^n - \left(\dfrac{r}{R}\right)^n\right] & r \geqslant \rho, \\ \dfrac{1}{2n}\left(\dfrac{r}{R}\right)^n\left[\left(\dfrac{R}{\rho}\right)^n - \left(\dfrac{\rho}{R}\right)^n\right] & r \leqslant \rho, \end{cases}$$

and the solution to the problem is

$$a_n(r) = \int_0^R G_n(r, \rho)\rho A_n(\rho)\,d\rho$$
$$= \frac{1}{2n}\left\{\int_0^r \left(\frac{\rho}{R}\right)^n\left[\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n\right]\rho A_n(\rho)\,d\rho + \int_r^R \left(\frac{r}{R}\right)^n\left[\left(\frac{R}{\rho}\right)^n - \left(\frac{\rho}{R}\right)^n\right]\rho A_n(\rho)\,d\rho\right\}$$
$$0 \leqslant r \leqslant R.$$

Solution to SAQ 4

Separation of variables for the corresponding homogeneous system leads to the boundary value problem

$$X'' + \lambda X = 0 \quad \text{in } (0, \pi),$$
$$X(0) = X(\pi) = 0,$$

which provides a suitable basis for expansion; the eigenfunctions are

$$\{\sin nx,\ n = 1, 2, \ldots\}.$$

Therefore we write

$$u(x, t) \sim \sum_1^\infty b_n(t)\sin nx,$$

where

$$b_n(t) = \frac{2}{\pi}\int_0^\pi u(x, t)\sin nx\, dx.$$

The finite Fourier transforms are given by

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin nx\, dx = \frac{2}{\pi}\left[\frac{\partial u}{\partial x}\sin nx - un\cos nx\right]_0^\pi - \frac{2}{\pi}n^2\int_0^\pi u\sin nx\, dx$$

$$= -n^2 b_n(t), \text{ using the boundary conditions;}$$

$$\frac{2}{\pi}\int_0^\pi \frac{\partial u}{\partial t}\sin nx\, dx = \frac{db_n(t)}{dt};$$

and

$$\frac{2}{\pi}\int_0^\pi \sin x\sin nx\, dx = \begin{cases} 1 & n = 1, \\ 0 & n \geqslant 2. \end{cases}$$

(sin x is its own Fourier Series, consisting of one term only.) Taking the finite sine transform of the differential equation we have

$$\frac{db_n}{dt} + n^2 b_n = \begin{cases} 1 & n = 1, \\ 0 & n \geqslant 2. \end{cases} \qquad (1)$$

The initial condition, with $t = 0$, gives

$$u(x, 0) = 1 \sim \sum_1 b_n(0)\sin nx.$$

The Fourier coefficients are given by

$$b_n(0) = \frac{2}{\pi}\int_0^\pi 1 \,.\, \sin nx\, dx = \begin{cases} 0 & n \text{ even}, \\ \dfrac{4}{n\pi} & n \text{ odd}, \end{cases}$$

and, in conjunction with (1), we obtain

$$b_1(t) = \frac{4}{\pi}e^{-t} + (1 - e^{-t}),$$

$$b_n(t) = 0 \quad n \text{ even},$$

$$b_n(t) = \frac{4}{n\pi}e^{-n^2 t} \quad n \text{ odd}, n > 1.$$

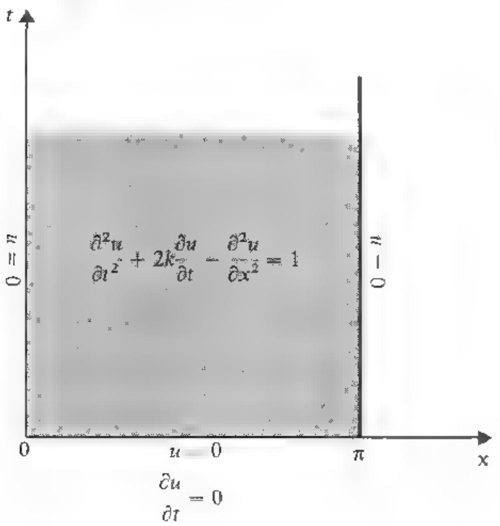
Then

$$u(x, t) = (1 - e^{-t})\sin x + \frac{4}{\pi}\sum_{n\text{ odd}}\frac{1}{n}e^{-n^2 t}\sin nx.$$

Solution to SAQ 5

As in the solution to SAQ 4, we expect to be able to use the expansion

$$u(x, t) \sim \sum_{n=1}^{\infty} b_n(t)\sin nx.$$



Accordingly, using the boundary conditions at $x = 0, \pi$, we obtain the finite sine transforms,

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin nx\,dx = -n^2 b_n,$$

$$\frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial t^2}\sin nx\,dx = \frac{d^2 b_n}{dt^2},$$

$$\frac{2}{\pi}\int_0^\pi \frac{\partial u}{\partial t}\sin nx\,dx = \frac{db_n}{dt},$$

$$\frac{2}{\pi}\int_0^\pi \sin nx\,dx = \begin{cases} 0 & n \text{ even,} \\ \dfrac{4}{n\pi} & n \text{ odd.} \end{cases}$$

Therefore, multiplying the differential equation by $\dfrac{2}{\pi}\sin nx$ and integrating, we obtain

$$\frac{d^2 b_n}{dt^2} + 2k\frac{db_n}{dt} + n^2 b_n = \begin{cases} 0 & n \text{ even,} \\ \dfrac{4}{n\pi} & n \text{ odd.} \end{cases}$$

The general solution is easily found to be

$$b_n(t) = e^{-kt}[A_n \cos\sqrt{(n^2 - k^2)}t + B_n \sin\sqrt{(n^2 - k^2)}t] + \begin{cases} 0 & n \text{ even,} \\ \dfrac{4}{n^3\pi} & n \text{ odd.} \end{cases}$$

The initial conditions yield

$$b_n(0) = \frac{db_n}{dt}(0) = 0,$$

whence

$$0 = A_n + \begin{cases} 0 & n \text{ even,} \\ \dfrac{4}{n^3\pi} & n \text{ odd,} \end{cases}$$

$$0 = -kA_n + \sqrt{(n^2 - k^2)}B_n.$$

Thus

$$A_n = B_n = 0 \qquad n \text{ even,}$$

$$A_n = -\frac{4}{n^3\pi}, \quad B_n = \frac{-4k}{n^3\pi\sqrt{(n^2 - k^2)}} \qquad n \text{ odd,}$$

and, finally

$$u(x, t) = -\frac{4}{\pi}\sum_{n\,\text{odd}} \frac{1}{n^3}\left\{e^{-kt}\left[\cos\sqrt{(n^2 - k^2)}t + \frac{k}{\sqrt{(n^2 - k^2)}}\sin\sqrt{(n^2 - k^2)}t\right] - 1\right\}\sin nx.$$

This problem represents a form of damped vibrations of a string driven by a uniformly distributed force along its length and starting from rest. Physically we would expect the vibrations to damp out and the string to settle into an equilibrium position as $t \to \infty$. Thus

$$\lim_{t\to\infty} u(x, t) = \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{1}{n^3}\sin nx$$

describes the "final" shape of the string. Since this is independent of t, we should also expect it to result from writing $\partial u/\partial t = 0$ in the original equation. Let

$$u_\infty : x \longmapsto \lim_{t\to\infty} u(x, t):$$

then u_∞ should satisfy

$$-\frac{d^2u_\infty}{dx^2} = 1,$$

$$u_\infty(0) = u_\infty(\pi) = 0.$$

Hence

$$\frac{du_\infty}{dx} = -x + A,$$

$$u_\infty = \tfrac{1}{2}x^2 + Ax + B,$$

and the boundary conditions give $A = \frac{1}{2}\pi$, $B = 0$, so that

$$u_\infty(x) = -\tfrac{1}{2}x(x - \pi).$$

We expand this as a sine series over $[0, \pi]$:

$$u_\infty(x) \sim \sum_{n=1}^{\infty} D_n \sin nx,$$

where

$$D_n = \frac{2}{\pi}\int_0^\pi -\tfrac{1}{2}x(x-\pi)\sin nx\, dx = -\left[\frac{2}{\pi n^3}\cos nx\right]_0^\pi$$

$$= \begin{cases} 0 & n \text{ even}, \\ \dfrac{4}{\pi n^3} & n \text{ odd}. \end{cases}$$

Hence,

$$u_\infty(x) = -\tfrac{1}{2}x(x - \pi) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^3}\sin nx,$$

as expected.

Solution to SAQ 6

As in SAQs 4 and 5 we take $\{\sin nx\}$ as a basis. If u is a solution we may expand it in a Fourier series as

$$u(x, y) \sim \sum_{n=1}^{\infty} b_n(y)\sin nx,$$

where

$$b_n(y) = \frac{2}{\pi}\int_0^\pi u(x, y)\sin nx\, dx.$$

The differential equation becomes, on transformation,

$$\frac{d^2b_n}{dy^2} - n^2b_n + b_n = \frac{d^2b_n}{dy^2} - (n^2 - 1)b_n = 0; \qquad (1)$$

the boundary conditions become

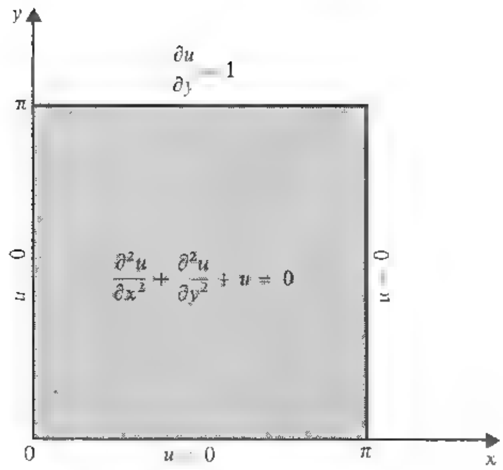
$$u(x, 0) = 0 = \sum_{n=1}^{\infty} b_n(0)\sin nx,$$

$$\frac{\partial u}{\partial y}(x, \pi) = 1 = \sum_{n=1}^{\infty} b_n'(\pi)\sin nx,$$

so that

$$b_n(0) = 0 \qquad \text{all } n,$$

$$b_n'(\pi) = \begin{cases} 0 & n \text{ even}, \\ \dfrac{4}{n\pi} & n \text{ odd}. \end{cases}$$



The general solution of (1) is

$$b_n(y) = A_n e^{y(n^2-1)^{\frac{1}{2}}} + B_n e^{-y(n^2-1)^{\frac{1}{2}}}$$

for $n = 2, 3, 4 \dots$, whilst

$$b_1(y) = A_1 + B_1 y.$$

The boundary conditions give us

$$A_n + B_n = 0 \qquad n = 2, 3, 4, \dots,$$

$$(n^2 - 1)^{\frac{1}{2}}[A_n e^{\pi(n^2-1)^{\frac{1}{2}}} - B_n e^{-\pi(n^2-1)^{\frac{1}{2}}}] = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n = 3, 5, 7, \dots; \end{cases}$$

in addition,

$$b_1(0) = A_1 = 0$$

and

$$b_1'(\pi) = B_1 = \frac{4}{\pi}.$$

Thus

$$A_n = B_n = 0 \qquad n \text{ even}$$

$$A_n = -B_n = \frac{2}{n\pi(n^2-1)^{\frac{1}{2}} \cosh \pi(n^2-1)^{\frac{1}{2}}} \qquad n = 3, 5, 7, \dots.$$

The solution is therefore given by

$$u(x, y) = \frac{4}{\pi} y \sin x + \frac{4}{\pi} \sum_{\substack{n > 1 \\ n \text{ odd}}} \frac{1}{n(n^2-1)^{\frac{1}{2}}} \frac{\sinh y(n^2-1)^{\frac{1}{2}}}{\cosh \pi(n^2-1)^{\frac{1}{2}}} \sin nx,$$

since the series may be shown to be uniformly convergent on $[0, \pi] \times [0, \pi]$.

Solution to SAQ 7

The basis for our expansion will be the eigenfunctions of the system

$$\alpha^2 \frac{d^2 X}{dx^2} + \lambda X = 0 \qquad x \in (0, a),$$

$$X(0) = X(a) = 0,$$

obtained by separating variables in the given equation. The functions

$$\sin \frac{\sqrt{\lambda}}{\alpha} x$$

are suitable, where, to satisfy $X(a) = 0$, we require

$$n\pi = \frac{\sqrt{\lambda}}{\alpha}a \qquad n - 1, 2, \ldots,$$

that is,

$$\lambda = \frac{\alpha^2 n^2 \pi^2}{a^2}.$$

Hence we put

$$u(x, t) \sim \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{a}$$

with

$$b_n(t) = \frac{2}{a}\int_0^a u(x, t)\sin\frac{n\pi x}{a}\,dx.$$

Then,

$$\frac{2}{a}\int_0^a \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{a}\,dx = -\frac{n^2\pi^2}{a^2}b_n,$$

$$\frac{2}{a}\int_0^a \frac{\partial u}{\partial t}\sin\frac{n\pi x}{a}\,dx = \frac{db_n}{dt}.$$

The equation becomes, under transformation,

$$\frac{db_n}{dt} = -\frac{n^2\pi^2\alpha^2}{a^2}b_n + \beta^2 b_n$$

or

$$\frac{db_n}{dt} - \left(\beta^2 - \frac{n^2\pi^2\alpha^2}{a^2}\right)b_n = 0,$$

for each $n \in Z^+$, which has the general solution

$$b_n(t) = B_n\exp\left[\left(\beta^2 - \frac{n^2\pi^2\alpha^2}{a^2}\right)t\right].$$

The coefficients B_n are determined by the initial conditions. In general, $B_1 \neq 0$ and it is immediately apparent that, if

$$\beta^2 - \frac{1^2\pi^2\alpha^2}{a^2} > 0,$$

the solution tends to infinity. (Note that as t increases, the first term dominates the series.) Thus the condition for the reaction to grow is

$$a > \frac{\pi\alpha}{\beta},$$

since $\alpha, \beta > 0$.

In physical terms, the coefficient β, which governs the rate of production of fresh neutrons, predominates over α, which governs the rate of diffusion out of the bar at its ends.

Solution to SAQ 8

We shall show that

$$h:(r, \theta) \longmapsto \log[f(r, \theta)]$$

is a solution of Laplace's equation wherever $f(r, \theta) \neq 0$, for the functions

$$f_1:(r, \theta) \longmapsto R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta - \phi)$$

and

$$f_2:(r,\theta)\longmapsto r^2+\rho^2-2r\rho\cos(\theta-\phi),$$

where ρ and ϕ are constants.

We have

$$\frac{\partial h}{\partial r}=\frac{1}{f}\frac{\partial f}{\partial r},$$

$$\frac{\partial^2 h}{\partial r^2}=\frac{1}{f}\frac{\partial^2 f}{\partial r^2}-\frac{1}{f^2}\left(\frac{\partial f}{\partial r}\right)^2$$

and

$$\frac{\partial^2 h}{\partial \theta^2}=\frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}-\frac{1}{f^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Hence

$$\nabla^2 h=\frac{1}{f}\nabla^2 f-\frac{1}{f^2}\left[\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right].$$

Now

$$\nabla^2[r\cos(\theta-\phi)]=0,$$

as we saw in Section 10.1.1. Let

$$f:(r,\theta)\longmapsto P+Qr^2-2r\rho\cos(\theta-\phi),$$

where P, Q are constants. Then

$$\nabla^2 h(r,\theta)=\frac{4Q}{f(r,\theta)}-\frac{4[Qr-\rho\cos(\theta-\phi)]^2+4\rho^2\sin^2(\theta-\phi)}{[f(r,\theta)]^2}$$

$$=\frac{4[PQ+Q^2r^2-2Qr\rho\cos(\theta-\phi)-Q^2r^2+2Qr\rho\cos(\theta-\phi)-\rho^2]}{[f(r,\theta)]^2}$$

$$=\frac{4[PQ-\rho^2]}{[f(r,\theta)]^2}.$$

For f_1 we have $P=R^2$ and $Q=\rho^2/R^2$; for f_2 we have $P=\rho^2$ and $Q=1$. In either case $PQ=\rho^2$ and so

$$\nabla^2 h=0.$$

Thus

$$\nabla^2 G(r,\theta;\rho,\phi)=\nabla^2\log f_1(r,\theta)+\nabla^2\log f_2(r,\theta)$$

$$=0\qquad\text{for } (r,\theta)\neq(\rho,\phi).$$

Solution to SAQ 9

We require a Green's function G such that, for each $(\rho,\phi)\in(0,R)\times(0,\pi)$,

$$G(r,\theta;\rho,\phi)=-\frac{1}{4\pi}\log[r^2+\rho^2-2r\rho\cos(\theta-\phi)]+\gamma(r,\theta;\rho,\phi),$$

where

$$\nabla^2\gamma=0\qquad(r,\theta)\in(0,R)\times(0,\pi)$$

and

$$G(R,\theta;\rho,\phi)=G(r,0;\rho,\phi)=G(r,\pi;\rho,\phi)=0.$$

Let $\bar{G}$ denote Green's function for the circle $0<r<R$ given by Equation (30.5) in *W*:

$$\bar{G}(r,\theta;\rho,\phi)=\frac{1}{4\pi}\left\{-\log[r^2+\rho^2-2r\rho\cos(\theta-\phi)]+\log\left[R^2+\frac{r^2\rho^2}{R^2}-2r\rho\cos(\theta-\phi)\right]\right\}.$$

Then

$$\nabla^2 \bar{G}(r, \theta : \rho, \phi) = 0 \qquad (r, \theta) \neq (\rho, \phi)$$

and

$$\nabla^2 \bar{G}(r, \theta : \rho, -\phi) = 0 \qquad (r, \theta) \neq (\rho, -\phi).$$

Thus the sum of $\bar{G}(r, \theta : \rho, \phi)$ and any multiple of $\bar{G}(r, \theta : \rho, -\phi)$ satisfies Laplace's equation at all points of the semicircle $0 < r < R$, $0 < \theta < \pi$ except $(r, \theta) = (\rho, \phi)$, and vanishes for $r = R$. We seek a suitable λ such that

$$\bar{G}(r, \theta : \rho, \phi) + \lambda \bar{G}(r, \theta : \rho, -\phi) = 0 \qquad \theta = 0, \pi.$$

Thus we require

$$\cos(-\phi) = -\lambda \cos\phi$$

and

$$\cos(\pi - \phi) = -\lambda \cos(\pi + \phi),$$

which are satisfied by $\lambda = -1$. Hence Green's function for the semicircle is given by

$$G(r, \theta : \rho, \phi) = \frac{1}{4\pi}\left\{-\log[r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)] + \log\left[R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta - \phi)\right]\right.$$
$$\left. + \log[r^2 + \rho^2 - 2r\rho\cos(\theta + \phi)] - \log\left[R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta + \phi)\right]\right\},$$

which is in the required form, by SAQ 8.

Solution to SAQ 10

We may proceed as in *W: Section 29*, to obtain the equations at the top of *W: page 130* with the additional boundary conditions

$$a_n(1) = b_n(1) = 0.$$

We must then construct the Green's function g_n given by

$$(rg_n')' - \frac{n^2}{r} g_n = 0 \qquad r \neq \rho,$$

$$g_n|_{r=1} = g_n|_{r=R} = 0,$$

$$g_n(r, \rho) = g_n(\rho, r),$$

$$g_n'|_{r=\rho+0} - g_n'|_{r=\rho-0} = -\frac{1}{\rho}.$$

These equations may be solved in a manner analogous to that used in SAQ 3.

When $n = 0$, the general solution of the homogeneous equation is

$$A \log r + B,$$

so

$$g_0(r, \rho) = \begin{cases} A_1(\rho)\log r & r < \rho, \\ A_2(\rho)\log\dfrac{R}{r} & r > \rho. \end{cases}$$

The symmetry condition gives

$$A_1(\rho) = A \log\frac{R}{\rho},$$

$$A_2(\rho) = A \log\rho,$$

and, since

$$g_0'|_{r=\rho+0} - g_0'|_{r=\rho-0} = -\frac{1}{\rho},$$

we have

$$-\frac{A\log\rho}{\rho}-\frac{A\log(R/\rho)}{\rho}=-\frac{1}{\rho},\qquad \text{i.e. } A=\frac{1}{\log R};$$

so that

$$g_0(r,\rho)=\begin{cases}\dfrac{\log r\log(R/\rho)}{\log R} & r\leqslant\rho,\\[2mm] \dfrac{\log\rho\log(R/r)}{\log R} & r\geqslant\rho.\end{cases}$$

For $n>0$, we obtain, in the same way

$$g_n(r,\rho)=\begin{cases}\dfrac{r^n-r^{-n}}{2n(R^n-R^{-n})}\left[\left(\dfrac{R}{\rho}\right)^n-\left(\dfrac{\rho}{R}\right)^n\right] & r\leqslant\rho,\\[2mm] \dfrac{\rho^n-\rho^{-n}}{2n(R^n-R^{-n})}\left[\left(\dfrac{R}{r}\right)^n-\left(\dfrac{r}{R}\right)^n\right] & r\geqslant\rho.\end{cases}$$

Arguing as in *W: Section 30* we find that

$$G(r,\theta;\rho,\phi)=\frac{1}{2\pi}g_0(r,\rho)+\sum_{n=1}^{\infty}\frac{1}{\pi}g_n(r,\rho)\cos n(\theta-\phi)$$

$$=\begin{cases}\dfrac{1}{2\pi}\left\{\dfrac{\log r\log(R/\rho)}{\log R}+\displaystyle\sum_{n=1}^{\infty}\dfrac{r^n-r^{-n}}{n(R^n-R^{-n})}\left[\left(\dfrac{R}{\rho}\right)^n-\left(\dfrac{\rho}{R}\right)^n\right]\cos n(\theta-\phi)\right\} & r<\rho,\\[2mm] \dfrac{1}{2\pi}\left\{\dfrac{\log\rho\log(R/r)}{\log R}+\displaystyle\sum_{n=1}^{\infty}\dfrac{\rho^n-\rho^{-n}}{n(R^n-R^{-n})}\left[\left(\dfrac{R}{r}\right)^n-\left(\dfrac{r}{R}\right)^n\right]\cos n(\theta-\phi)\right\} & r>\rho.\end{cases}$$

An alternative method of solution is to proceed as in *W: page 136*. We write

$$G=\gamma^*+\gamma$$

where

$$\gamma^*(r,\theta;\rho,\phi)=-\frac{1}{4\pi}\log[\rho^2+r^2-2\rho r\cos(\theta-\phi)]$$

$$=\begin{cases}-\dfrac{1}{2\pi}\left[\log\rho-\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{r}{\rho}\right)^n\cos n(\theta-\phi)\right] & r<\rho,\\[2mm] -\dfrac{1}{2\pi}\left[\log r-\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{\rho}{r}\right)^n\cos n(\theta-\phi)\right] & r>\rho.\end{cases}$$

To find γ we solve, by separation of variables, the problem

$$\nabla^2\gamma=0\qquad\text{in the annulus } 1<r<R,$$

$$\gamma(1,\theta:\rho,\phi)=\frac{1}{2\pi}\left[\log\rho-\sum_{n=1}^{\infty}\frac{1}{n}\rho^{-n}\cos n(\theta-\phi)\right],$$

$$\gamma(R,\theta:\rho,\phi)=\frac{1}{2\pi}\left[\log R-\sum_{n=1}\frac{1}{n}\left(\frac{\rho}{R}\right)^n\cos n(\theta-\phi)\right],$$

$$\gamma(r,\theta.\rho,\phi)=(r,\theta+2\pi:\rho,\phi).$$

We then obtain the series for G given above.

Solution to SAQ 11

Proceeding as in SAQ 10, we look at the system

$$(rg_n')' - \frac{n^2}{r} g_n = 0 \qquad r \neq \rho,$$

$$g_n|_{r \to 0} \text{ is bounded, } g_n' + g_n|_{r=R} = 0,$$

$$g_n(r, \rho) = g_n(\rho, r),$$

$$g_n'|_{r=\rho+0} - g_n'|_{r=\rho-0} = -\frac{1}{\rho}.$$

When $n = 0$, the general solution of the differential equation is

$$A \log r + B,$$

so to satisfy the boundary conditions we require

$$g_0(r, \rho) = \begin{cases} A_1(\rho) & r < \rho, \\ A_2(\rho)\left[\log\left(\dfrac{r}{R}\right) - \dfrac{1}{R}\right] & r > \rho. \end{cases}$$

The symmetry condition gives

$$A_1(\rho) = A\left[\log\left(\frac{\rho}{R}\right) - \frac{1}{R}\right],$$

$$A_2(\rho) = A,$$

and we determine the constant A by the discontinuity in the derivative,

$$A\left[\frac{1}{\rho}\right] = -\frac{1}{\rho}, \qquad \text{i.e. } A = -1.$$

Thus

$$g_0(r, \rho) = \begin{cases} \dfrac{1}{R} + \log\dfrac{R}{\rho} & r \leqslant \rho, \\ \dfrac{1}{R} + \log\dfrac{R}{r} & r \geqslant \rho. \end{cases}$$

For $n > 0$, the general solution is

$$Ar^n + Br^{-n}$$

so to satisfy the boundary conditions we require

$$g_n(r, \rho) = \begin{cases} A_n(\rho) r^n & r < \rho, \\ B_n(\rho)\left[\left(\dfrac{n-R}{n+R}\right)\left(\dfrac{r}{R}\right)^n + \left(\dfrac{R}{r}\right)^n\right] & r > \rho. \end{cases}$$

Also the symmetry condition gives

$$A_n(\rho) = C\left[\left(\frac{n-R}{n+R}\right)\left(\frac{\rho}{R}\right)^n + \left(\frac{R}{\rho}\right)^n\right], \qquad B_n(\rho) = C\rho^n,$$

and the jump condition gives

$$Cn\left\{\rho^n\left[\left(\frac{n-R}{n+R}\right)\frac{\rho^{n-1}}{R^n} - \frac{R^n}{\rho^{n+1}}\right] - \left[\left(\frac{n-R}{n+R}\right)\left(\frac{\rho}{R}\right)^n + \left(\frac{R}{\rho}\right)^n\right]\rho^{n-1}\right\} = -\frac{1}{\rho};$$

so that

$$g_n(r, \rho) = \begin{cases} \dfrac{1}{2n(n+R)}\left[(n-R)\left(\dfrac{\rho}{R}\right)^n + (n+R)\left(\dfrac{R}{\rho}\right)^n\right]\left(\dfrac{r}{R}\right)^n & r \leqslant \rho, \\ \dfrac{1}{2n(n+R)}\left(\dfrac{\rho}{R}\right)^n\left[(n-R)\left(\dfrac{r}{R}\right)^n + (n+R)\left(\dfrac{R}{r}\right)^n\right] & r \geqslant \rho. \end{cases}$$

Following the argument of *W: Section 30*, we have

$$G(r, \theta : \rho, \phi) = \frac{1}{2\pi} g_0(r, \rho) + \sum_{n=1}^{\infty} \frac{1}{\pi} g_n(r, \rho) \cos n(\theta - \phi)$$

$$= \begin{cases} \dfrac{1}{2\pi}\left\{\dfrac{1}{R} + \log\dfrac{R}{\rho} + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+R)}\left[(n-R)\left(\frac{\rho}{R}\right)^n + (n+R)\left(\frac{R}{\rho}\right)^n\right]\left(\frac{r}{R}\right)^n \cos n(\theta - \phi)\right\} & r < \rho, \\ \dfrac{1}{2\pi}\left\{\dfrac{1}{R} + \log\dfrac{R}{r} + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+R)}\left(\frac{\rho}{R}\right)^n\left[(n-R)\left(\frac{r}{R}\right)^n + (n+R)\left(\frac{R}{r}\right)^n\right] \cos n(\theta - \phi)\right\} & r > \rho. \end{cases}$$

Solution to SAQ 12

The element of integration in plane polar coordinates is $dA = r\,dr\,d\theta$, so that

$$\delta(\mathbf{r} - \boldsymbol{\rho}) = \frac{\delta(r - \rho)\delta(\theta - \psi)}{r}.$$

For $\rho = 0$,

$$\delta(\mathbf{r}) = \frac{\delta(r)}{2\pi r}.$$

For, integrating over a domain D which contains the origin, we have

$$\iint_D f(r, \theta) \frac{\delta(r)}{2\pi r} r\,dr\,d\theta = \frac{1}{2\pi} f(0) \int_0^{2\pi} d\theta$$

$$= f(0) = \iint_D f(\mathbf{r})\delta(\mathbf{r})\,dA,$$

which is the correct result.

Solution to SAQ 13

Let

$$u(x) = \int_{-\infty}^{\infty} H(x, \xi) f(\xi)\,d\xi,$$

and write L for the linear operator

$$L : u \longmapsto (pu')' + qu,$$

so that

$$L[u](x) = L\int_{-\infty}^{\infty} H(x, \xi) f(\xi)\,d\xi$$

$$= \int_{-\infty}^{\infty} L[H](x, \xi) f(\xi)\,d\xi$$

$$= -\int_{-\infty}^{\infty} \delta(x - \xi) f(\xi)\,d\xi$$

$$= -f(x) \qquad x \in R.$$

Thus, irrespective of the boundary conditions on H, we obtain a particular solution for u. (Note that this justifies our argument in note (iv) of Section 10.1.2.)

Solution to SAQ 14

$$u(\mathbf{r}) = \int_D G(\mathbf{r} : \boldsymbol{\rho}) f(\boldsymbol{\rho})\,d\boldsymbol{\rho}$$

so that

$$L[u]\boldsymbol{\rho} = L\int_D G(\mathbf{r} : \boldsymbol{\rho}) f(\boldsymbol{\rho})\,d\boldsymbol{\rho}$$

$$= \int_D L[G](\mathbf{r} ; \boldsymbol{\rho}) f(\boldsymbol{\rho})\,d\boldsymbol{\rho}$$

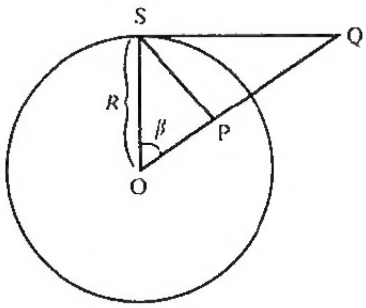
$$= -\int_D \delta(\mathbf{r} - \boldsymbol{\rho}) f(\boldsymbol{\rho})\,d\boldsymbol{\rho}$$

$$= -f(\mathbf{r}) \qquad \mathbf{r} \in D.$$

Solution to SAQ 15

Using the cosine law, where OP $= p$, OQ $= q$, we have

$$(SP)^2 = R^2 + p^2 - 2Rp\cos\beta,$$
$$(SQ)^2 = R^2 + q^2 - 2Rq\cos\beta.$$



Eliminating $\cos\beta$ we obtain

$$\frac{(SQ)^2}{q} - \left(\frac{R^2}{q} + q\right) = \frac{(SP)^2}{p} - \left(\frac{R^2}{p} + p\right).$$

Now $pq = R^2$, so substituting for R^2,

$$\frac{(SQ)^2}{q} - (p + q) = \frac{(SP)^2}{p} - (q + p).$$

Hence,

$$\frac{SP}{SQ} = \left(\frac{p}{q}\right)^{\frac{1}{2}} = \frac{p}{R}$$

which is independent of β and SP/SQ does not vary as S describes the circumference of the circle.

Solution to SAQ 16

Take P to be the point (ρ, ϕ), so that Q is $(R^2/\rho, \phi)$, and take T to be (r, θ), and, by SAQ 15,

$$\frac{QT}{PT} = \frac{R}{\rho}$$

when T is on the boundary. Then

$$G(r, \theta : \rho, \phi) = \frac{1}{2\pi}\left[\log\frac{QT}{PT} - \log\frac{R}{\rho}\right]$$

satisfies the equation

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} = -\delta(\mathbf{r} - \rho)$$

and vanishes on $r = R$. Therefore

$$G(r, \theta; \rho, \phi) = \frac{1}{4\pi}\left\{\log\frac{\rho^2}{R^2}(QT)^2 - \log(PT)^2\right\}$$

$$= -\frac{1}{4\pi}\log[r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)] + \frac{1}{4\pi}\log\left[R^2 + \frac{r^2\rho^2}{R^2} - 2r\rho\cos(\theta - \phi)\right]$$

as found in *W: Section 30.*

Solution to SAQ 17

See *W: page 139, lines −10 to −7.*

Solution to SAQ 18

The boundary of the domain is $x = 0$, so we have, using the formula given in the text,

$$u(x, y) = -\int_{-\infty}^{\infty} \frac{\partial G}{\partial n}(x, y : \xi, \eta) f(\eta)\, d\eta.$$

Remember we need the directional derivative along the *outward* normal, so

$$\frac{\partial G}{\partial n} = -\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0} = -\frac{\partial}{\partial \xi}\left\{\frac{1}{4\pi}\log\left[\frac{(x+\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y-\eta)^2}\right]\right\}_{\xi=0}$$

$$= \frac{-x}{\pi[x^2 + (y-\eta)^2]}.$$

Then,

$$u(x, y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(\eta)\, d\eta}{[x^2 + (y-\eta)^2]}.$$

PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

1	*W*	The Wave Equation
2	*W*	Classification and Characteristics
3	*W*	Elliptic and Parabolic Equations
4		NO TEXT
5	*S*	Finite-Difference Methods I: Initial Value Problems
6	*W*	Fourier Series
7	*N*	Motion of Overhead Electric Train Wires
8	*S*	Finite-Difference Methods II: Stability
9	*W*	Green's Functions I: Ordinary Differential Equations
10	*W*	Green's Functions II: Partial Differential Equations
11	*S*	Finite-Difference Methods III: Boundary Value Problems
12		NO TEXT
13	*W*	Sturm-Liouville Theory
14	*W*	Bessel Functions
15	*N*	Finite-Difference Methods IV: Parabolic Equations
16	*N*	Blood Flow in Arteries

The letter after the unit number indicates the relevant set book; N indicates a unit not based on either book.

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conducting bodies, and is an immediate consequence of what has preceded. For let x, y, z, be the rectangular co-ordinates of any particle p in the interior of one of the bodies; then will $-\left(\frac{dV}{dx}\right)$ be the force with which p is impelled in the direction of the co-ordinate x, and tending to increase it. In the same way $-\frac{dV}{dy}$ and $-\frac{dV}{dz}$ will be the forces in y and z, and since the fluid is in equilibrium all these forces are equal to *zero*: hence

$$0 = \frac{dV}{dx}dx + \frac{dV}{dy}dy + \frac{dV}{dz}dz = dV,$$

which equation being integrated gives

$$V = \text{const.}$$

This value of V being substituted in the equation (1) of the preceding number gives

$$\rho = 0,$$

and consequently shows, that the density of the electricity at any point in the interior of any body in the system is equal to *zero*.

The same equation (1) will give the value of ρ the density of the electricity in the interior of any of the bodies, when there are not perfect conductors, provided we can ascertain the value of the potential function V in their interior.

(3.) Before proceeding to make known some relations which exist between the density of the electric fluid at the surfaces of bodies, and the corresponding values of the potential functions within and without those surfaces, the electric fluid being confined to them alone, we shall in the first place, lay down a general theorem which will afterwards be very useful to us. This theorem may be thus enunciated:

Let U and V be two continuous functions of the rectangular co-ordinates x, y, z, whose differential co-efficients do not become infinite at any point within a solid body of any form whatever; then will

$$\int dxdydz\,U\delta V + \int d\sigma\,U\left(\frac{dV}{dw}\right) = \int dxdydz\,V\delta U + \int d\sigma\,V\left(\frac{dU}{dw}\right);$$

the triple integrals extending over the whole interior of the body, and those relative to $d\sigma$, over its surface, of which $d\sigma$ represents an element: dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body.

To prove this let us consider the triple integral

$$\int dxdydz\left\{\left(\frac{dV}{dx}\right)\left(\frac{dU}{dx}\right)+\left(\frac{dV}{dy}\right)\left(\frac{dU}{dy}\right)+\left(\frac{dV}{dz}\right)\left(\frac{dU}{dz}\right)\right\}.$$

The method of integration by parts, reduces this to

$$\int dydz\,V''\frac{dU''}{dx} - \int dydz\,V'\frac{dU'}{dx} + \int dxdz\,V''\frac{dU''}{dy} - \int dxdz\,V'\frac{dU'}{dy}$$
$$+\int dxdy\,V''\frac{dU''}{dz} - \int dxdy\,V'\frac{dU'}{dz} - \int dxdydz\,V\left\{\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right\};$$

the accents over the quantities indicating, as usual, the values of those quantities at the limits of the integral, which in the present case are on the surface of the body, over whose interior the triple integrals are supposed to extend.

Let us now consider the part $\int dydz\,V''\frac{dU''}{dx}$ due to the greater values of x. It is easy to see since dw is every where perpendicular to the surface of the solid, that if $d\sigma''$ be the element of this surface corresponding to $dydz$, we shall have

$$dydz = -\frac{dx}{dw}d\sigma'',$$

and hence by substitution

$$\int dydz\,V''\frac{dU''}{dx} = -\int d\sigma''\frac{dx}{dw}V''\frac{dU''}{dx}.$$

In like manner it is seen, that in the part $-\int dydz\,V'\frac{dU'}{dx}$ due to the smaller values of x, we shall have $dydz = +\frac{dx}{dw}d\sigma'$, and consequently

$$-\int dydz\,V'\frac{dU'}{dx} = -\int d\sigma'\frac{dx}{dw}V'\frac{dU'}{dx}.$$

Then, since the sum of the elements represented by $d\sigma'$, together with those represented by $d\sigma''$, constitute the whole surface of the body, we have by adding these two parts

$$\int dydz\left(V''\frac{dU''}{dx} - V'\frac{dU'}{dx}\right) = -\int d\sigma\frac{dx}{dw}V\frac{dU}{dx};$$

where the integral relative to $d\sigma$ is supposed to extend over the whole surface, and dx to be the increment of x corresponding to the increment dw.

In precisely the same way we have

$$\int dxdz\left(V''\frac{dU''}{dy} - V'\frac{dU'}{dy}\right) = -\int d\sigma\frac{dy}{dw}V\frac{dU}{dy},$$

$$\text{and}\int dxdy\left(V''\frac{dU''}{dz} - V'\frac{dU'}{dz}\right) = -\int d\sigma\frac{dz}{dw}V\frac{dU}{dz};$$

therefore, the sum of all the double integrals in the expression before given will be obtained by adding together the three parts just found; we shall thus have

$$-\int d\sigma V\left\{\frac{dU}{dx}\frac{dx}{dw}+\frac{dU}{dy}\frac{dy}{dw}+\frac{dU}{dz}\frac{dz}{dw}\right\} = -\int d\sigma V\frac{dU}{dw};$$

where V and $\frac{dU}{dw}$ represent the values at the surface of the body. Hence, the integral

$$\int dxdydz\left\{\frac{dV}{dx}\frac{dU}{dx}+\frac{dV}{dy}\frac{dU}{dy}+\frac{dV}{dz}\frac{dU}{dz}\right\},$$

by using the characteristic δ in order to abridge the expression, becomes

$$-\int d\sigma V\frac{dU}{dw} - \int dxdydz\,V\delta U.$$

Poisson, in the commencement of his first memoir (Mem. de l'Institut 1811), has incidentally given a method for determining the distribution of electricity on the surface of a spheroid of any form, which would naturally present itself to a person occupied in these researches, being in fact nothing more than the ordinary one noticed in our introductory observations, as requiring the resolution of the equation *(a)*. Instead however of supposing, as we have done, that the point p must be upon the surface, in order that the equation may subsist, M. Poisson availing himself of a general fact, which was then supported by experiment only, has conceived the equation to hold good wherever this point may be situated, provided it is within the spheroid, but even with this extension the method is liable to the same objection as before.

Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, and reflecting on the advantages that arise in the solution of many difficult problems, from dispensing altogether with a particular examination of each of the forces which actuate the various bodies in any system, by confining the attention solely to that peculiar function on whose differentials they all depend, I was induced to try whether it would be possible to discover any general relations, existing between this function and the quantities of electricity in the bodies producing it. The advantages Laplace had derived in the third book of the Mecanique Celeste, from the use of a partial differential equation of the second order, there given, were too marked to escape the notice of any one engaged with the present subject, and naturally served to suggest that this equation might be made subservient to the object I had in view. Recollecting, after some attempts to accomplish it, that previous researches on partial differential equations, had shown me the necessity of attending to what have, in this Essay, been denominated the singular values of functions, I found, by combining this consideration with the preceding, that the resulting method was capable of being applied with great advantage to the electrical theory, and was thus, in a short time, enabled to demonstrate the general formulæ contained in the preliminary part of the Essay. The remaining part ought to be regarded principally as furnishing particular examples of the use of these general formulæ; their number might with great ease have been increased, but those which are given, it is hoped, will suffice to point out to mathematicians, the mode of applying the preliminary results to any case they may wish to investigate. The hypotheses on which the received theory of magnetism is founded, are by no means so certain as the facts on which the electrical theory rests; it is however not the less necessary to have the means of submitting them to calculation, for the only way that appears open to us in the investigation of these subjects, which seem as it were desirous to conceal themselves from our view, is to form the most probable hypotheses we can, to deduce rigorously the consequences which flow from them, and to examine whether such consequences agree numerically with accurate experiments.

The applications of analysis to the physical Sciences, have the double advantage of manifesting the extraordinary powers of this wonderful instrument of thought, and at the same time of serving to increase them; numberless are the instances of the truth of this assertion. To select one we may remark, that M. Fourier, by his investigations relative to heat, has not only discovered the general equations on which its motion depends, but has likewise been led to new analytical formulæ, by whose aid M. M. Cauchy & Poisson have been enabled to give the complete theory of the motion of the waves in an indefinitely

extended fluid. The same formulæ have also put us in possession of the solutions of many other interesting problems too numerous to be detailed here.—It must certainly be regarded as a pleasing prospect to analists, that at a time when astronomy, from the state of perfection to which it has attained, leaves little room for farther applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it; and, amongst other things, probably the theory that supposes light to depend on the undulations of a luminiferous fluid, and to which the celebrated Dr. T. Young has given such plausibility, may furnish a useful subject of research, by affording new opportunities of applying the general theory of the motion of fluids. The number of these opportunities can scarcely be too great, as it must be evident to those who have examined the subject, that, although we have long been in possession of the general equations on which this kind of motion depends, we are not yet well acquainted with the various limitations it will be necessary to introduce, in order to adapt them to the different physical circumstances which may occur.

Should the present Essay tend in any way to facilitate the application of analysis to one of the most interesting of the physical sciences, the author will deem himself amply repaid for any labour he may have bestowed upon it; and it is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man, who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded.

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